Solutions to exercise 2a of Long Test 1

Exercise 2a) Describe by a system of linear equations the subspace $W \subseteq \mathbb{R}^5$ with basis given by $\{(1,1,1,1,1),(0,0,1,1,0)\}$.

Solution 1

This solution follows the method shown in class.

We consider the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$. The subspace W is the linear span of the rows of the matrix.[1] We bring the matrix to reduced echelon form, obtaining $\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$. The row vectors of this matrix are also a basis for W, and from them we can get a (minimal) set of equations describing W. Indeed,

$$\begin{aligned} W &= \{a(1,1,0,0,1) + b(0,0,1,1,0) : a,b \in \mathbb{R}\} \\ &= \{(a,a,b,b,a) : a,b \in \mathbb{R}\} \end{aligned}$$

from which we see that W can be described as the subspace of solutions (x_1,x_2,x_3,x_4,x_5) to the following system of linear equations:

$$\begin{cases} x_2 = x_1 \\ x_4 = x_3 \\ x_5 = x_1 \end{cases} \quad \text{which we may rewrite as} \quad \begin{cases} x_2 - x_1 = 0 \\ x_4 - x_3 = 0 \\ x_5 - x_1 = 0 \end{cases}$$

Remark 0.1. The method above always works in this type of exercises thanks to the properties of the reduced echelon form. We can express all of the non-pivot coordinates (above, x_2, x_4 and x_5) in terms of the pivots (x_1 and x_3).

Solution 2

Here's an alternative solution. It's longer and more roundabout, so I wouldn't recommend it, but you can use it if you prefer it, of course.

We consider the auxiliary subspace of $\mathbb{R}^5$ given by

$$E = \{(a_1,a_2,a_3,a_4,a_5) \in \mathbb{R}^5 : a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0 \text{ for all } (x_1,x_2,x_3,x_4,x_5) \in W\}.$$

This is the subspace of all the 5-tuples of coefficients of homogeneous equations which are satisfied by all the vectors in W. We have a basis for W consisting of $w_1 = (1,1,1,1,1)$ and $w_2 = (0,0,1,1,0)$, so we can simplify E as follows:

$$E = \{(a_1,a_2,a_3,a_4,a_5) \in \mathbb{R}^5 : a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0 \text{ for } (x_1,x_2,x_3,x_4,x_5) \in \{w_1,w_2\}\}.$$

which, in other words, says that E is the subspace of solutions to the homogeneous system of linear equations represented by $\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$, whose reduced echelon form is $\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$. Therefore, E can be expressed as the subspace of solutions to the system

[1] This is sometimes called the *row space* of the matrix.

of equations

$$\begin{cases} a_1 = -a_2 - a_5 \\ a_3 = -a_4 \end{cases} \tag{0.2}$$

with free variables given by $a_2, a_4, a_5 \in \mathbb{R}$. In other words,

$$\begin{aligned} E &= \{(-a_2 - a_5, a_2, -a_4, a_4, a_5) : a_2, a_4, a_5 \in \mathbb{R}\} \\ &= \operatorname{lin}((-1,1,0,0,0),(0,0,-1,1,0),(-1,0,0,0,1)). \end{aligned}$$

We now return to the meaning of E: its vectors give us the coefficients for homogeneous equations satisfied by all vectors of W. We have found a basis for E consisting of three vectors, so the three equations they furnish us are enough to describe W. In conclusion, W is the subspace of solutions of the following homogeneous system of linear equations:

$$\begin{cases} -x_1 + x_2 = 0 \\ -x_3 + x_4 = 0 \\ -x_1 + x_5 = 0 \end{cases}. \tag{0.3}$$

Remark 0.4. Do not confuse E with W! Many students gave the system of equations describing E as the final answer to the question. That is incorrect. Note that W has dimension 2 whereas E has dimension 3. Note as well the choice of coordinate names above: we used x_i for the coordinates of the vectors in W, and a_i for the coordinates of the vectors in E. This helps not to confuse them. Some of you wrote both systems (0.3) and (0.2) with the same variable names and not making any commentary, making it look as if they were both describing W.

Here's a simpler example, for emphasis. Consider $W = \operatorname{lin}((1,0),(0,1))$; we actually have $W = \mathbb{R}^2$. That's the span of the horizontal vectors in the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. On the other hand, we could consider the subspace E of solutions to the homogeneous system of equations encoded by the matrix $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$. The system of equations is $\begin{cases} x = 0 \\ y = 0 \end{cases}$, so actually $E = \{0\}$. That's very different from W!

For a more in-depth description of the general relation between subspaces like E and W, see the next section.

More advanced remarks for the curious

Can we describe the relation between W and E more precisely? Is it a coincidence that their dimensions sum to the total dimension of the space they're in, namely 5?

There's a useful device that we can use here, and that is the scalar product $\langle v, w \rangle$, sometimes also written $v \cdot w$, and which may be familiar to you already. [2] It takes two vectors from $\mathbb{R}^n$ and gives a real number. In $\mathbb{R}^n$, the definition is

$$\langle (a_1, \ldots, a_n), (x_1, \ldots, x_n) \rangle = a_1 x_1 + \cdots + a_n x_n \in \mathbb{R}.$$

[2] Don't confuse it with the similarly-named "multiplication by a scalar" αv where $\alpha \in \mathbb{R}$.

So our subspace $E \subseteq \mathbb{R}^5$ in Solution 2 above is actually

$$E = \{v \in \mathbb{R}^5 : \langle v, w \rangle = 0 \text{ for all } w \in W\} =: W^\perp.$$

The notation $W^\perp$ is defined exactly as above: it is the subspace of vectors of the ambient space (here, $\mathbb{R}^5$) whose scalar product with all vectors of W is 0. It's called the *perpendicular subspace* for W; two vectors v and w are called *perpendicular* if $\langle v, w \rangle = 0$, and all this has a geometrical interpretation. (Can you geometrically describe the perpendicular subspace to the plane $z = 0$ in $\mathbb{R}^3$, for example?) This gives one answer to the first question above: E is the perpendicular subspace to W.

It can be proven that W and $W^\perp$ are always complementary, in the sense that $W \cap W^\perp = \{0\}$ and moreover any vector can be decomposed as a (unique) sum of two vectors, one in W and one in $W^\perp$.[3] These conditions imply that, for $W \subseteq \mathbb{R}^n$, we have $\dim W + \dim W^\perp = n$, answering the second question posed above: it's not a coincidence, it's a feature of perpendicular subspaces.

As a final remark using the concept of a kernel of a linear transformation, let us note the following. If we have a basis $\{w_1, \dots, w_k\}$ of $W \subseteq \mathbb{R}^n$, then in fact the condition that the vectors in $W^\perp$ satisfy can be checked only in the basis vectors,

$$\begin{aligned} W^\perp &= \{v \in \mathbb{R}^n : \langle v, w_i \rangle = 0 \text{ for all } i = 1, \dots, k\} \\ &= \ker(T) \end{aligned}$$

where $T : \mathbb{R}^n \to \mathbb{R}^k$ is the linear map given by $T(v) = (\langle v, w_1 \rangle, \cdots, \langle v, w_k \rangle)$.

Notice that $A := M(T)^{\text{st}}_{\text{st}}$ is the $k \times n$ matrix whose rows are exactly the w_i. This implies that $\ker(T) = \ker A$, where $\ker(A) := \{X \in \mathbb{R}^n : AX = 0\}$ is the kernel of A, also called its *null space*. Putting all of this together, we have sketched a proof of the following

Proposition. *Let $W \subseteq \mathbb{R}^n$ be a subspace with basis $\mathcal{B} = \{w_1, \dots, w_k\}$. Let A be the matrix whose rows are the vectors of $\mathcal{B}$. Then* $\ker(A) = W^\perp$.

Similarly, we have the following proposition: we don't even need the rows to be linearly independent to get a statement.

Proposition. *The null space and the row space of a matrix are perpendicular.*

Proof. Suppose the matrix is called A and it has dimensions $k \times n$. Let $w_1, \dots, w_k$ denote the row vectors of A. Similarly as before, notice that for any $X \in \mathbb{R}^n$,

$$AX = \begin{pmatrix} \langle w_1, X \rangle \\ \vdots \\ \langle w_k, X \rangle \end{pmatrix},$$

so $AX = 0$ if and only if X is perpendicular to the row vectors of A which span its row space. $\square$

[3]For $W \subseteq \mathbb{R}^n$, this is summarized by saying that $\mathbb{R}^n$ is the *direct sum* of W and $W^\perp$, and written $\mathbb{R}^n = W \oplus W^\perp$.