Fiber bundles and characteristic classes

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Abstract

This is a very quick introduction to the theory of fiber bundles and characteristic classes, with an emphasis on Stiefel-Whitney and Chern classes. Some caveats are in order: given that this is intended to fit in a two- to three-hour lecture, many things have been sacrificed. There are no proofs, few examples, very possibly missing hypotheses (I invite you to assume all base spaces are paracompact, or why not just work in the category of CW-complexes), and some blackboxes along the way. The interested reader should leaf through the references.

Some conventions:

Space will mean "topological space" and a *map* between these will mean "continuous map".

K will denote either of $\mathbb{R}$ or $\mathbb{C}$, endowed with their usual topology.

1 Fiber bundles

1.1 General fiber bundles

The main definition is the following:

Definition 1.1.1. A *fiber bundle with total space E, base space B, projection p and fiber F* is a quadruple (E, B, p, F) where $p : E \to B$ is a map which is *locally trivial*, meaning that for all $b \in B$ there exists an open set $U \subset B$ with $b \in U$ and a homeomorphism $\varphi : p^{-1}(U) \to U \times F$ (called a *local trivialization*) such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "p"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd} \tag{1}$$

where p_1 is the projection onto the first coordinate.

We also write $F \to E \xrightarrow{p} B$ to denote (E, B, p, F). This is not an unreasonable notation in view of the following remark.

Remark 1.1.2. The map φ restricts to a homeomorphism $p^{-1}(b) \xrightarrow{\cong} \{b\} \times F \cong F$. We can thus call every $p^{-1}(b)$ a fiber of the bundle.

Example 1.1.3.

1. The *trivial bundle with base space B and fiber F*: take $p = p_1 : B \times F \to B$.

2. A fiber bundle with discrete fiber is the same thing as a covering space, if the base is path-connected.

3. The Möbius strip is the total space of a fiber bundle with base space the circle, and projection the retraction onto it.

4. Let M be a smooth n-manifold and let TM be the *tangent bundle*, i.e. the union of all the tangent spaces:
$$TM = \bigsqcup_{x \in M} T_xM = \{(x, v) : x \in M, v \in T_xM\}.$$
Then TM is a smooth manifold, and it is the total space of a fiber bundle with base space M, projection $p : TM \to M, (x, v) \mapsto x$ and fiber $\mathbb{R}^n \cong T_pM$ for all x.

   We can also consider the *cotangent bundle*, where instead of T_xM one considers the dual vector space T_xM^*, or more generally, the bundle of *alternating k-tensors* if we consider $\Lambda^k(T_xM^*)$ for a fixed $k \geq 1$ (cf. differential forms).

   If the manifold is Riemannian (e.g. if it is embedded in an ambient Euclidean space and we let the tangent spaces inherit the inner product of the ambient vector space), then we can consider a *normal bundle NM* where we consider the orthogonal complements $T_xM^\perp$.

5. Is there a diffeomorphism $TS^2 \to S^2 \times \mathbb{R}^2$ such that $h(T_xS^2) = \{x\} \times \mathbb{R}^2$? This is what it means to ask whether TS^2 is trivial. The answer is no, for if we had such a diffeomorphism, then the vector field X on S^2 defined as $X(x) = h^{-1}(x, v)$, for a fixed $v \neq 0$, would be a nowhere vanishing vector field on S^2, which is not possible by the classical "hairy ball theorem".

Exercice 1.1.4. Prove that TS^1 is trivial, using the fact that $S^1 \subset \mathbb{C}$ which is a field. You can also prove that TS^3 and TS^7 are trivial, by means of Hamilton's quaternions and of octonions.

1.2 Fiber bundles with structure group

Suppose we have two local trivializations φ, φ' over $U \subset B$: then the map $\varphi \circ \varphi'^{-1} : U \times F \to U \times F$ has the form
$$(u, f) \mapsto (u, \theta_{\varphi,\varphi'}(u)(f))$$
for some function $\theta_{\varphi,\varphi'} : U \to \mathrm{Homeo}(F)$ called a *transition function*. A priori $\theta_{\varphi,\varphi'}$ needs merely land in the set of functions $F \to F$, but thanks to remark 1.1.2 it really lands in $\mathrm{Homeo}(F)$.

Now, consider for example the case of the tangent bundle TM. The fiber is $F = \mathbb{R}^n$ which is a vector space. We could consider only the transition functions $\theta_{\varphi,\varphi'} : U \to GL_n(\mathbb{R}) \subset \text{Homeo}(\mathbb{R}^n)$ to get coherence with respect to this additional linear structure. This is the basis for one possible definition of a vector bundle.

Let us do this in general. Instead of considering $\mathbb{R}^n$ and the group $GL_n(\mathbb{R})$, we will consider any space F with the action of a topological group G.

Definition 1.2.1. A *topological group* is a group G which is also a space, in such a way that the product $G \times G \to G$ and the inversion $G \to G$ are continuous maps. One sometimes calls an old-fashioned group a *discrete group*.

Let X be a space and G be a topological group. An *action* of G on X is a usual group-theoretic action $G \times X \to X$ that is a continuous map. In this case we say that X is a G*-space*.

Remark 1.2.2.

1. The condition of continuity in the definition of an action is equivalent to having the adjoint group homomorphism $G \to \text{Bij}(X)$ have its image contained in the subgroup $\text{Homeo}(X)$, where $\text{Bij}(X)$ is the group of bijections of X.

2. We can endow $\text{Homeo}(X)$ with the compact-open topology. Then, for locally compact Hausdorff spaces, the data of an action of G on X is equivalent to the data of a continuous group homomorphism $G \to \text{Homeo}(X)$.

Example 1.2.3. $GL_n(\mathbb{R})$ is a topological group. So is its subgroup $O(n)$ of linear automorphisms of $\mathbb{R}^n$ preserving the inner product. They act on $\mathbb{R}^n$ by evaluation.

We will also want to consider the complex case: $GL_n(\mathbb{C})$ is a topological group, and so is its subgroup $U(n)$ of linear automorphisms of $\mathbb{C}^n$ preserving the Hermitian inner product on $\mathbb{C}$. They act on $\mathbb{C}^n$ by evaluation.

Definition 1.2.4. Let G be a topological group acting on a space F. Let (E, B, p, F) be a fiber bundle. A G*-atlas* for the bundle is a collection $\{(U_i, \varphi_i)\}$ where $\{U_i\}$ is an open covering of B and $\varphi_i : p^{-1}(U_i) \to U_i \times F$ is a local trivialization, such that: if $U_i \cap U_j \neq \emptyset$, then

$$\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

is given by

$$(u, f) \mapsto (u, \theta_{i,j}(u) \cdot f)$$

where $\theta_{i,j} : U_i \cap U_j \to G$ is a continuous map, called *transition function*.

Two G-atlases are *equivalent* if their union is a G-atlas.

A *fiber bundle with structure group* G, or a G*-bundle*, is the data of: a fiber bundle where the fiber has an action of G, and an equivalence class of a G-atlas. [1]

Remark 1.2.5. A fiber bundle (E, B, p, F) with structure group $\text{Homeo}(F)$ is the same thing as a fiber bundle (E, B, p, F).

[1] Compare with the definition of a smooth structure on a topological manifold, and with that of a smooth manifold.

1.3 Vector bundles

Definition 1.3.1. A *K-vector bundle of rank n* is a fiber bundle with fiber K^n and structure group $GL_n(K)$. An alternative name is *n-plane bundle*; if $n = 1$, we talk of *line bundles*. If $p : E \to B$ is such a bundle, we denote $\dim(E) := n$.

Remark 1.3.2. The data of a K-vector bundle of rank n (E, B, p, F) is equivalent to the data of a map of spaces $p : E \to B$ satisfying the following:

- for every $b \in B$ the subspace $p^{-1}(b) \subset E$ is a K-vector space of dimension n,
- the map $p : E \to B$ is locally trivial and the local trivializations restrict to linear isomorphisms on the fibers. More precisely, for every $b \in B$ there is an open neighboorhood $U \subset B$ and a homeomorphism $\varphi : p^{-1}(U) \to U \times K^n$ such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "p"'] & & U \times K^n \arrow[dl, "p_1"] \\ & U & \end{tikzcd} \tag{2}$$

  commutes, and such that the restrictions $\varphi : p^{-1}(b') \to \{b'\} \times K^n \xrightarrow{\cong} K^n$ are linear isomorphisms for every $b' \in U$.

Example 1.3.3. The examples in 1.1.3.4 are all vector bundles.

1.4 Principal bundles

A G-principal bundle is a particularly simple fiber bundle with structure group G, and actually, any such fiber bundle can be constructed out of a principal one.

Definition 1.4.1. Let G be a topological group. A *G-principal bundle* is a fiber bundle with fiber G and structure group G, where G acts on itself by left translations.

Remark 1.4.2. The condition that the structure group be G with the self-action by left translations is equivalent to the action of G on the fiber F being free and transitive. Indeed, observe that the action of a group G on a space F is free and transitive if and only if the map $\varphi_x : G \to X$, $g \mapsto g \cdot x$ is a G-isomorphism, for every $x \in X$.

Example 1.4.3. Recall that a covering space $p : E \to B$ over a path-connected base B is called *regular* if the covering transformation group $\mathrm{A}ut_p(E)$ acts transitively on the fibers, or equivalently, $p_*(\pi_1(E)) \subset \pi_1(B)$ is a normal subgroup. In this case, we can identify the deck transformation group as follows:

$$G := \mathrm{A}ut_p(E) \cong \pi_1(B)/p_*(\pi_1(E)).$$

Given that this action is always free[2], we can identify the fiber of p with G, and the action is by left translations.

[2]Recall that an action of G on X is *free* if: if $g \cdot x = x$ for some x, then g is the identity. Equivalently, stabilizers are trivial.

In conclusion, a regular covering space over a path-connected base $p : E \to B$ is a $\pi_1(B)/p_*(\pi_1(E))$-principal bundle. In particular, a universal covering space $\tilde{B} \to B$ is a $\pi_1(B)$-principal bundle.

The previous example generalizes as follows. We will not make use of this proposition, but we state it since we think it might shed some light on principal bundles:

Proposition 1.4.4. *If $p : E \to B$ is a G-principal bundle, then G acts freely on E on the right, and $E/G \cong B$.*

One could carefully construct a statement saying that the converse to the previous proposition is true, e.g., a G-principal bundle is the same thing as a fiber bundle of the form $p : E \to E/G$, the quotient map, where p is a fiber bundle with structure group G and G acts freely on the right on E. One has to be careful, there is this and other very similar-looking definitions of principal bundles on the literature.

Remark 1.4.5. Let $p : E \to B$ be a G-principal bundle and F be a G-space. Then one can make a *Borel construction* $E \times_G F$ with a projection $E \times_G F \to B$ giving a fiber bundle with fiber F and structure group G: this bundle and the one determined by p have the same transition maps. Conversely, from any fiber bundle with structure group G one can obtain a principal G-bundle. (cf. [1, (4.5)]).

We will not need to use the details of these constructions. So we take the previous paragraph as the necessary mathematical blackbox needed for confidently asserting the following: when studying fiber bundles with structure group G, we can, for simplicity, restrict to the study of G-principal bundles, without losing information.

In particular, for vector bundles, we can restrict to the study of $GL_n(K)$-principal bundles. But we can further restrict our attention. There is a notion of *reduction of the structure group*: sometimes, given a subgroup $H \subset G$, it happens that studying G-principal bundles is the same as studying H-principal bundles, as any G-principal bundle can be "reduced" to an H-principal bundle. Once more, we prefer to take this as a blackbox. We will also admit that $GL_n(\mathbb{R})$ can be reduced to $O(n)$ and $GL_n(\mathbb{C})$ can be reduced to $U(n)$. Thus,

> *The study of $\mathbb{R}$-vector bundles (resp. $\mathbb{C}$-vector bundles) of rank n is equivalent to the study of $O(n)$- (resp. $U(n)$-) principal bundles.*

Evidence for this is given by the fact that $O(n) \subset GL_n(\mathbb{R})$, $U(n) \subset GL_n(\mathbb{C})$ are deformation retracts (think Gram-Schmidt). See [5, (11.44), (11.45)] for details.

1.4.1 Hopf bundles

Let us introduce a very classical family of examples of principal bundles. Consider the quotient map from a sphere to real, projective or quaternionic projective space. It is an exercise to prove that they are the projections of principal bundles as follows:

$$S^0 \longrightarrow S^n \longrightarrow \mathbb{R}P^n$$

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$$

$$S^3 \longrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n$$

Take $n = 1$. We obtain the following principal bundles:

$$\begin{gathered} S^0 \longrightarrow S^1 \longrightarrow S^1 \\ S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2 \\ S^3 \longrightarrow S^7 \longrightarrow S^4 \end{gathered} \tag{3}$$

There is an additional one coming from the octonionic projective line:

$$S^7 \longrightarrow S^{15} \longrightarrow S^8 \tag{4}$$

All of these bundles are sometimes called "Hopf bundles", but if one is to talk about *the* Hopf bundle then η is the one classically called thusly.

Theorem 1.4.6 (Adams)**.** *The bundles from (3) and (4) are the only fiber bundles with base space, total space and fiber being spheres.*

This is a very deep theorem. It is related to the theorem I will now state. Recall exercise 1.1.4: you proved that TS^1, TS^3 and TS^7 were trivial. Of course, TS^0 is also trivial. Also recall that in example 1.1.3.5 we observed that TS^2 is not trivial.

Theorem 1.4.7 (Adams)**.** *The only tangent bundles to spheres which are trivial are TS^0, TS^1, TS^3 and TS^7.*

Another related theorem:

Theorem 1.4.8 (Adams)**.** *The only real division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.*

This is a remarkable instance where a deep algebraic theorem follows from topological techniques, instead of the more usual other way around.

While we're dealing with these bundles, let me know list some fun facts that you can safely ignore if you're unacquainted with homotopy theory:

- $\pi_3(S^2)$ is a free abelian group of rank 1, generated by η. This was historically surprising and first proven by Hopf (in our modern language, it follows from the long exact sequence of homotopy groups associated to the Hopf fibration). With homology, one has that the homology groups of rank greater than the dimension of the sphere vanish. Mathematicians were expecting this to be true also for homotopy groups: it seemed to them to be the reasonable thing. It was a shock when Hopf proved otherwise.

- $\pi_4(S^3)$ is the free $\mathbb{Z}/2$-module of rank 1 generated by $\Sigma\eta$, the suspension of η. The proof of this requires more involved techniques, such as e.g. the Serre spectral sequence.

- $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ for $n \geq 3$, generated by the corresponding iterated suspension of η. This is a particular case of Freudenthal's suspension theorem, the starting point for stable homotopy theory.

1.5 Classifying spaces

To avoid cluttering, we have not discussed morphisms between bundles yet. We now introduce the concept in the generality we need:

Definition 1.5.1. Let G be a topological group and $p' : E' \to B$, $p : E \to B$ be two G-principal bundles over the same base space B. A *morphism* from p' to p is given by a map $f : E' \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E' \arrow[rr, "f"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

and such that the restrictions $f : p^{-1}(b) \to q^{-1}(b)$ are G-maps for every $b \in B$.

This determines a category of G-principal bundles over B. We might want to study B by looking at this category. For example, for the particular case of vector bundles, this idea leads to the development of topological K-theory, on which we will not delve here.

Proposition 1.5.2. *Let $p : E \to B$ be a G-principal bundle and $f : B' \to B$ be a map. There exists a G-principal bundle $f^*E \to B'$, called the* pullback of p, *and a map $f^*E \to E$ making the following diagram a pullback diagram of spaces.*

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

Now for something important: every G-principal bundle can be obtained as a pullback from a fixed G-principal bundle.

Theorem 1.5.3. *Let G be a topological group. Any G-principal bundle $EG \to BG$ with EG a contractible space satisfies the following: for any paracompact space X,[3] the map*

$$[X, BG] \to \{\text{isomorphism classes of } G\text{-principal bundles over } X\}$$

that sends $[f]$ to the isomorphism class of the pullback $f^*(EG)$, is a bijection. Here

- *$[X, BG]$ denotes the set of homotopy classes of maps $X \to BG$,*
- *the space BG is called a* classifying space *of G and is unique up to homotopy equivalence,*
- *the G-principal bundle $EG \to BG$ is called a* universal G-principal bundle, *and*
- *a map $f_p : X \to BG$ whose homotopy class corresponds to the isomorphism class of a G-principal bundle p over X is called a* classifying map *of p; it is unique up to homotopy.*

[3] Dear Reader, I trust you will be lenient towards this sudden change of notation conventions: henceforth, our base spaces will be called X, since the letter B typically denotes classifying spaces.

There exists such a bundle.

In view of the uniqueness statement, we will talk of "the" classifying space, and "the" universal bundle.

In other words, the theorem says that if $p : E \to X$ is a G-principal bundle, then there exists a unique-up-to-homotopy map $f_p : X \to BG$ such that there is a pullback diagram as follows:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & EG \arrow[d] \\ X \arrow[r, "f_p"'] & BG \end{tikzcd}$$

This equates the study of G-principal bundles to the study of $[X, BG]$ for various X, i.e., to a problem in homotopy theory. This is the starting point for the theory of characteristic classes.

A couple of words on a nice heuristic. There is the trivial bundle which is just a product. Other bundles which are not isomorphic to the trivial one display some "twisting": for example, the Möbius strip. In a very hand-wavy sense, all the twisting is already there in the universal bundle: the twisting of any bundle is encoded in it, and we decode it through the classifying map. For example, the classifying map is trivial if and only if the bundle is trivial.

Now, some words about BG. There are several ways to construct it. One can prove that we are in the correct hypotheses to apply the Brown representability theorem whose conclusion gives what we want. One could alternatively use a bar construction for a model of BG which is in addition functorial, an often useful fact.

Remark 1.5.4. Let G be a discrete group. Then BG is an Eilenberg-Mac Lane space of type $K(G, 1)$, i.e., a space with only one non-trivial homotopy group, which is G in degree 1.

The proof is easy but has some homotopy theory prerequisites. It goes like this: as in example 1.1.3.2, we get that $EG \to BG$ is a covering map. Therefore it is an isomorphism in homotopy groups starting from 2. Combining this with contractibility of EG and using the long exact homotopy sequence of the fibration $G \to EG \to BG$, we get the result.

Example 1.5.5. By the previous remark, we immediately get that $B\mathbb{Z} = S^1$ and $B\mathbb{Z}/2 = \mathbb{R}P^\infty$. We could construct those directly, though. For the first example: $\mathbb{Z}$ acts on the real line $\mathbb{R}$ by translations. This is a free action and $\mathbb{R}$ is contractible, which gives $E\mathbb{Z} = \mathbb{R}$ and $B\mathbb{Z} = E\mathbb{Z}/\mathbb{Z} \cong S^1$. For the second example, take the projection $S^\infty \to \mathbb{R}P^\infty$.

1.5.1 The universal vector bundles

We are especially interested in the case of vector bundles, so we are led to consider the classifying spaces of the groups $O(n)$, $U(n)$. Let us denote by $\mathrm{Vect}_n^K(X)$ the set of isomorphism classes of K-vector bundles over X of rank n.

For $n = 1$ it is easy to give a description of $BO(1)$. Indeed, $O(1) = S^0 \cong \mathbb{Z}/2$, and so the universal $O(1)$-principal bundle is $S^\infty \to \mathbb{R}P^\infty$ (example 1.5.5) hence

$$BO(1) = \mathbb{R}P^\infty \quad \text{and} \quad [X, \mathbb{R}P^\infty] \cong \mathrm{Vect}_1^{\mathbb{R}}(X).$$

In a similar fashion, one gets that the universal $U(1) \cong S^1$-principal bundle is the map $S^\infty \to \mathbb{C}P^\infty$ obtained by taking the colimit of the quotient maps $S^{2n+1} \to \mathbb{C}P^n$. Hence

$$BU(1) = \mathbb{C}P^\infty \quad \text{and} \quad [X, \mathbb{C}P^\infty] \cong \mathrm{Vect}_1^{\mathbb{C}}(X).$$

We could go on and give explicit models for $BO(n)$, $BU(n)$ for any n. We decline to do so.

What about the actual *line bundles*? The bundle $S^\infty \to \mathbb{R}P^\infty$ has fiber $O(1) = \mathbb{Z}/2$: it is a principal $O(1)$-bundle, which *corresponds* to a real line bundle (which in particular has fiber $\mathbb{R}$) *which we haven't described*. One can give an explicit model for this: it is the so-called *tautological line bundle over* $\mathbb{R}P^\infty$, obtained as the colimit of the tautological line bundles over $\mathbb{R}P^\ell$, which we denote γ_1^ℓ. Their total space is $E_1^\ell := \{(\ell, v) \in \mathbb{R}P^\ell \times \mathbb{R}^{\ell+1} : v \in \ell\}$, and the projections are $(\ell, v) \mapsto \ell$.

An analogous story holds for complex line bundles.

2 Characteristic classes

Let G be a topological group. Suppose we have two G-principal bundles over X, let us call them $p, q : E \to X$, and we want to decide whether they are isomorphic. Theorem 1.5.3 says we have classifying maps $f_p, f_q : X \to BG$. The bundles p and q are isomorphic if and only if f_p and f_q are homotopic. We haven't progressed much: deciding whether two maps are homotopic is very difficult!

However, the functors of algebraic topology come to our help. We can consider a cohomology theory h^* and the induced maps

$$h^*(f_p), h^*(f_q) : h^*(BG) \to h^*(X).$$

If we managed to find one such h^* such that $h^*(f_p) \neq h^*(f_q)$, then p and q would be non-isomorphic, and the problem would be solved.

Given an element $x \in h^*(BG)$ we can consider $x(p) := h^*(f_p)(x)$ and $x(q) := h^*(f_q)(x)$ in the cohomology of the base space. Such elements will be what we will call *characteristic classes* of p and q with respect to h^*, which is omitted from the notation. Finding an h^* such that $h^*(f_p) \neq h^*(f_q)$ is equivalent to finding an element x such that $x(p) \neq x(q)$.

The goal is to say some words on some well-understood families of characteristic classes on vector bundles, the Stiefel-Whitney classes and the Chern classes. The former live on singular cohomology with coefficients in $\mathbb{Z}/2$ and apply to real vector bundles, whereas the latter live on integral singular cohomology and apply to complex vector bundles.

Definition 2.0.6. Let G be a topological group and $p : E \to X$ be a G-principal bundle. Let h^* be a cohomology theory on topological spaces. A *characteristic class* is an element

$$x(p) := h^*(f_p)(x) \in h^*(X),$$

where $x \in h^*(BG)$.

Remark 2.0.7. There is a contravariant functor k_G from the homotopy category of topological spaces to the category of sets, assigning to a space X the set of isomorphism classes of G-principal bundles over X (on arrows, it is defined via the pullback construction). It is immediate to check that an element $x \in h^*(BG)$ defines a natural transformation from k_G to h^*:

$$\begin{tikzcd} h\mathbf{Top}^{\mathrm{op}} \arrow[r, bend left, "k_G"] \arrow[r, bend right, "h^*"'] \arrow[r, phantom, "\Downarrow x"] & \mathbf{Set} \end{tikzcd} .$$

One can ask whether all such natural transformations define characteristic classes. The answer is affirmative by theorem 1.5.3 which asserts that the functor k_G is representable and the Yoneda lemma: $\mathrm{Nat}(k_G, h^*) = \mathrm{Nat}([-, BG], h^*) \cong h^*(BG)$.

As we already said, our main interest will be the about $G = O(n)$, corresponding to rank n real vector bundles, and $G = U(n)$, corresponding to rank n complex vector bundles.

2.1 Stiefel-Whitney and Chern classes

We will not actually build them. We will state existence and uniqueness theorems characterizing them.

Theorem 2.1.1 (Stiefel-Whitney classes)**.** *There exists a unique sequence of functions $w_0, w_1, \dots$ which to each real vector bundle $E \to X$ associate a characteristic class $w_i(E) \in H^i(X; \mathbb{Z}/2)$, depending only on the isomorphism class of the bundle, such that, for every E,*

1. $w_0(E) = 1$,
2. $w_i(f^*E) = f^*(w_i(E)) \in H^i(Y; \mathbb{Z}/2)$ *if* $f : Y \to X$,
3. *Cartan's formula:* $w_n(E_1 \oplus E_2) = \sum_{i+j=n} w_i(E_1) \smile w_j(E_2)$ *for all* n *and all* E_1, E_2,[4]
4. $w_i(E) = 0$ *if* $i > \dim E$,
5. *If* $\gamma : L \to \mathbb{R}P^\infty$ *is the universal real line bundle, then* $w_1(\gamma) \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$ *is a generator.*

The classes $w_1(E), w_2(E), \dots$ *are called the* Stiefel-Whitney classes *of* E.

[4]I haven't discussed the direct sum of two vector bundles. It's easy: fiberwise, it's the usual direct sum of vector spaces. Of course, one has to really construct this bundle. Fiberwise constructions yield honest bundles for *smooth functors*.

Remark 2.1.2. In case you forgot, an easy way of computing $H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)=\mathbb{Z}/2$ is via the cellular chain complex.

Similarly, we have the Chern classes:

Theorem 2.1.3 (Chern classes)**.** *There exists a unique sequence of functions $c_0, c_1, \dots$ which to each complex vector bundle $E \to X$ associate a characteristic class $c_i(E) \in H^{2i}(X;\mathbb{Z})$, depending only on the isomorphism class of the bundle, such that, for every E,*

1. $c_0(E) = 1$,
2. $c_i(f^*E) = f^*(c_i(E)) \in H^{2i}(Y;\mathbb{Z})$ *if* $f : Y \to X$,
3. *Cartan's formula:* $c_n(E_1 \oplus E_2) = \sum_{i+j=n} c_i(E_1) \smile c_j(E_2)$ *for all n and all E_1, E_2,*
4. $c_i(E) = 0$ *if* $i > 2\dim E$,
5. *If $\gamma : L \to \mathbb{C}P^\infty$ is the universal complex line bundle, then $c_1(\gamma) \in H^2(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}$ is a generator.*

The classes $c_1(E), c_2(E), \dots$ are called the Chern classes *of E.*

Remark 2.1.4. Analogous remark as in 2.1.2 about computing $H^2(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}$.

A lot can be said about these characteristic classes. For lack of time, we will restrict to some basic facts. Before getting to examples, let me state a nice result:

Theorem 2.1.5.

1. *We have an isomorphism*

$$H^*(BO(n);\mathbb{Z}/2) = \mathbb{Z}/2[w_1, \dots, w_n]$$

where w_i is the i-th Stiefel-Whitney class of the universal real n-plane bundle and $|w_i| = i$. Thus any characteristic class of a real n-plane bundle is a polynomial on the Stiefel-Whitney classes of the universal n-plane bundle.

In particular, for $n = 1$,

$$H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[w_1].$$

2. *We have an isomorphism*

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_n]$$

where c_i is the i-th Chern class of the universal complex n-plane bundle, and $|c_i| = 2i$. Thus any characteristic class of a complex n-plane bundle is a polynomial on the Chern classes of the universal n-plane bundle.

In particular, for $n = 1$,

$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[c_1].$$

This was to be expected, in a way: if every vector bundle can be obtained from a universal one, then it's reasonable that every characteristic class for them is obtained from the Stiefel-Whitney and Chern classes, in view of their axiom 5.

Suppose we computed all the Stiefel-Whitney classes of a real vector bundle $p : E \to X$ of rank n, and they all vanish. By the previous theorem, all the characteristic classes in mod 2 singular cohomology are zero, which is the same as saying that the induced map

$$f_p^* : H^*(BO(n); \mathbb{Z}/2) \to H^*(X; \mathbb{Z}/2)$$

is zero.

Is this enough to guarantee that f_p is homotopic to a constant map, i.e., that p is isomorphic to a trivial vector bundle? No, it is not. We will see an example where Stiefel-Whitney classes are not sufficient in a second (but see 2.1.8.3 if you are impatient). First, another positive result: they *are* sufficient for line bundles!

Theorem 2.1.6. *$w_1 : \mathrm{Vect}_1^{\mathbb{R}}(X) \to H^1(X; \mathbb{Z}/2)$ and $c_1 : \mathrm{Vect}_1^{\mathbb{C}}(X) \to H^2(X; \mathbb{Z})$ are isomorphisms of abelian groups. Here Vect_1^K is an abelian group under the tensor product.*[5] *In particular, this says that*

$$w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$$

for real line bundles L_1, L_2, and similarly with Chern classes.

This theorem says several things. There's the formulas, which prove useful and important (e.g. in defining the Chern character). And as we said, there's the fact that two line bundles are isomorphic if and only if they have the same Stiefel-Whitney or Chern classes. But remember, this is a particularity of *line* bundles.

Let us finally have some computations. First, a little convention:

Definition 2.1.7. Define $w(E) = 1 + w_1(E) + w_2(E) + \cdots \in H^*(X; \mathbb{Z}/2)$, the *total Stiefel-Whitney class*. Also define the *total Chern class* by an analogous formula.

These simplify the Cartan formula: $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$, and analogously for the Chern classes.

Example 2.1.8.

1. A trivial bundle has total Stiefel-Whitney or Chern class equal to 1. Indeed: first, observe that the trivial bundle over the point, $* \times \mathbb{R}^n \to *$ has $w_i(* \times \mathbb{R}^n) \in H^i(*) = 0$ if $i > 0$, hence $w(* \times \mathbb{R}^n) = 1$. Now observe that every trivial bundle $X \times \mathbb{R}^n$ is a pullback of the trivial bundle over the point. Hence by axiom 2, we get the result.

2. Stiefel-Whitney and Chern classes are *stable*, in the sense that they remain the same after direct summing a trivial bundle. Indeed, this is a consequence of Cartan's formula and of the previous example.

[5]Once more, I haven't defined the tensor product of vector bundles, but you can guess: it's obtained fiberwise.

3. First observe that NS^n, the normal bundle of $S^n \subset \mathbb{R}^{n+1}$, is trivial. Indeed, consider $S^n \times \mathbb{R} \to NS^n, (x,t) \mapsto tx$.

   Now, let TS^n be the tangent bundle to the sphere S^n. Consider $TS^n \oplus NS^n$. On one hand, it is TS^n, as NS^n is trivial. On the other hand, it is also trivial: indeed, just sum both orthogonal vectors to get an isomorphism with the trivial bundle of rank $n+1$. So by the first example, $w(TS^n) = 1$.

   So all Stiefel-Whitney classes of TS^n vanish. Does this mean that TS^n is a trivial bundle? No. Recall Adams' theorem 1.4.7: TS^n is trivial only for $n = 0, 1, 3, 7$.

4. If L_n is the tautological line bundle over $\mathbb{R}P^n$, then $w(L_n) = 1 + u \in H^*(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2[u]/(u^{n+1})$. Indeed, this bundle can be obtained via pullback of the universal line bundle γ with respect to the inclusion map $j : \mathbb{R}P^n \to \mathbb{R}P^\infty$. But the map

$$j^* : H^1(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2 \to \mathbb{Z}/2 = H^1(\mathbb{R}P^n; \mathbb{Z}/2)$$

   is an isomorphism, hence $w_1(L_n) = j^* w_1(\gamma)$ is a generator of the right hand side.

And thus this quick introduction comes to an end. I would have liked to have said a word on the splitting principle. This is a theorem that many times allows us to reduce questions on general vector bundles to line bundles. One can deduce from this that, in a sense, one need only define the first Chern class of the universal line bundle, and then all Chern classes of all complex vector bundles are determined. This is a nice observation, and one that allows us to define a meaningful "theory of Chern classes" in more general contexts (cf. complex oriented cohomology theories).

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