A brief introduction to homotopy and homology

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These are informal notes aimed to introduce math people not into algebraic topology to a couple of our basic objects of study. I have made prominent use of footnotes, where I have stacked digressions, technicalities and comments for people who already have a background in the subject.

- The aim of algebraic topology is to study topological spaces via algebra.

  A coherent way to attach invariants to an object is a *functor*. Indeed, they have the essential property that any functor F preserves isomorphisms. So if $F(X)$ and $F(Y)$ are not isomorphic, X and Y cannot be isomorphic.

1 Into homotopy theory

- Our starting example is the path connected components of a space as a functor $\pi_0 : \mathrm{Spaces} \to \mathrm{Sets}$. But of course this is not a complete invariant, meaning that two spaces with same π_0 might not be homeomorphic (e.g. $\mathbb{R}$ and S^1: one is compact, the other isn't). [1] So we look for other, better invariants.

  We have: $\pi_0(X) = X/\sim$ where $x \sim y$ if there exists a (continuous)[2] path from x to y. A path in X can be seen as a homotopy between two maps from a one-pointed space to X. We want to generalize this, taking maps from objects higher-dimensional than a point. To do this correctly, we need to introduce basepoints.

- Let (X, x_0) be a pointed space.

  Consider $S^0 = \{-1, 1\}$ the unit sphere in $\mathbb{R}$. Denote by $\mathrm{Map}_*(S^0, X)$ the set of pointed maps $(S^0, -1) \to (X, x_0)$. Then $\mathrm{Map}_*(S^0, X)$ is in bijection with X.

  We define $\sim$ in $\mathrm{Map}_*(S^0, X)$ as: $f \sim g$ if there is a path from $f(1)$ to $g(1)$. This gives a bijection between $\pi_0(X)$ and $\mathrm{Map}_*(S^0, X)/\sim$.

  But now we remark that $\sim$ in $\mathrm{Map}_*(S^0, X)$ is exactly the based homotopy relation: $f \sim g$ if there exists a map $H : S^0 \times [0,1] \to X$ such that $H(-, 0) = f$ and $H(-, 1) = g$ (H is a homotopy) and $H(-1, -) = x_0$ (H is *based*: the basepoint doesn't move).

  Thus $\pi_0(X) = [S^0, X]$. Here if S and X are based spaces, $[S, X]$ denotes $\mathrm{Map}_*(S, X)$ modulo the based homotopy relation.

[1] Technically speaking, the functor π_0 does not *reflect isomorphisms*.

[2] From now on every map between spaces will be assumed to be continuous.

- We are thus led to define functors $\pi_n : \text{Spaces}_* \to \text{Sets}$ as:

$$\pi_n(X, x_0) = [S^n, X]$$

where S^n has had a point fixed as basepoint.[3] If we have a based map $f : X \to Y$, this induces a map $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ as $\pi_n(f)([\gamma]) = [f \circ \gamma]$.

For example, $\pi_1(X)$ is the set of based homotopy classes of loops in X based in x_0.

- We can now distinguish some spaces that we couldn't with just π_0. For example, take $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{(0,0)\}$, both with some chosen basepoint, say $(1,0)$. They are both path-connected so their π_0 is a point. But their π_1 is different: $\pi_1(\mathbb{R}^2)$ has only one element (consider linear homotopies) but $\pi_1(\mathbb{R}^2 \setminus \{(0,0)\})$ has non-trivial loops that go around the origin. [4]

- Now, it happens that the $\pi_n(X)$ for $n \geq 1$ are more than sets: they're groups, and as such they are usually called the *homotopy groups* of X; the induced maps are group homomorphisms. For π_1, the operation amounts to concatenation of loops. The structure comes from the "pinch" map that spheres have, $S^n \to S^n \vee S^n$, if $n \geq 1$. It happens that for $n \geq 2$, $\pi_n(X)$ is always abelian.

So $\pi_0 : \text{Spaces}_* \to \text{Sets}$, $\pi_1 : \text{Spaces}_* \to \text{Groups}$,[5] and $\pi_n : \text{Spaces}_* \to \text{Abelian Groups}$, for $n \geq 2$.

- There is an important remark to be made. The π_n being functors, they will not distinguish between homeomorphic spaces. It seems acceptable that we want to classify spaces *up to homeomorphism*. But actually, there is another equivalence relation, coarser than homeomorphism, such that two equivalent spaces are being identified by our π_n.

We say that X and Y are *homotopy equivalent* if there are maps $f : X \to Y$, $g : Y \to X$ such that gf and fg are both *homotopic* to the identity. [6] Such spaces have identical π_n's.

More generally, if $f, g : X \to Y$ are homotopic, then $\pi_n(f) = \pi_n(g)$ for all n. This is simply the fact that if $\alpha : S^n \to X$ and $f \simeq g$, then $f \circ \alpha \simeq g \circ \alpha$.

- A moment of manifold reflection can lead us to think this is too coarse a tool. A fundamental invariant of a smooth manifold is its dimension, and this invariant is not being preserved by the π_n's. For example, $\mathbb{R}^m$ and the point both have all their π_n's trivial ($\mathbb{R}^n$ and the point are not homeomorphic but are homotopy equivalent).

The "answer" is that, yes, the homotopy groups are not that great *geometric* invariants, but they're still ok for some purposes. In fact, they are *perfect* invariants of the *homotopy type* of a space, which justifies the name "homotopy group":

Theorem (Whitehead, 1949). *If a map $f : X \to Y$ between CW complexes induces an isomorphism in all of the π_n's, then f is a homotopy equivalence.*

[3] I will be sloppy and denote $\pi_n(X, x_0)$ by $\pi_n(X)$.

[4] This example fuels the slogan that "homotopy groups can detect holes". This is in a certain sense true, but subtle.

[5] They can be non-abelian. Ask me about $SO(3)$, or see page 184 of Stillwell's "Naive Lie Theory".

[6] This can be hard to visualize geometrically. A nice fact is that two spaces are homotopy equivalent if and only if they are both deformation retractions of a single space (a mapping cylinder). See Lee, Introduction to Topological Manifolds, proposition 7.46.

Here a *CW complex* is a special kind of topological space which is especially nice and easy to handle (e.g. it is normal (hence Hausdorff) and first countable), but also sufficiently broad. Any smooth manifold is homotopy equivalent to a CW complex, for instance.

From now on, we accept that what we are interested in is spaces *up to homotopy equivalence* instead of homeomorphism.[7] An equivalence class of spaces under homotopy equivalence is called a *homotopy type*.[8]

- Are we finished? Given Whitehead's theorem, we may think that we are done since we found perfectly powerful invariants. But with great power comes great incalculability.

2 Into (ordinary) homology

- We must look for some new invariants, more amenable to computation.

  In geometric topology, it has been known for a long time that a very useful tool to study a manifold is a triangulation.[9] This means, essentially, a decomposition of your manifold into triangles (or shapes homeomorphic to these).

  Of course, up to homeomorphism, a filled triangle is the same thing as a disk, but the added advantage is the important combinatorial structure behind triangles: we have edges, faces, vertices...

  Let us denote by Δ^n the *standard n-simplex*, i.e. the convex hull of the canonical vectors $\{e_0, \dots, e_n\} \subset \mathbb{R}^{n+1}$.

  Now, given a space X, what we will do is look at *all* (continuous) maps $\Delta^n \to X$. This gives a set denoted $\mathrm{Sing}_n(X)$ (for *singular*) for every $n \geq 0$.[10]

  We now wish to exploit the structure of triangles. We have $n+1$ *face maps* $\Delta^{n-1} \to \Delta^n$ satisfying some relations.

  By precomposition with these maps we obtain $n+1$ face maps $d_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X)$.

- What can we do with this Sing gadget? Take again the example of $\mathbb{R}^2$ versus $U = \mathbb{R}^2 \setminus \{(0,0)\}$: we wish to detect the "hole". Then one intuitive way in which triangles will be of help is the following: we can put three Δ^1's closing in around the origin. These three line segments bound a full Δ^2 in $\mathbb{R}^2$, but this is not the case in U.

[7] If you insist with manifolds, I should say that there is quite a bit of geometric topology constructions that end up in spaces-up-to-homotopy-type, e.g. in surgery theory, or in cobordism.

[8] A mild digression: a (metrizable) topological space can be defined as an equivalence class of metric spaces, under the relation "mutual metric ball containment". Then one realizes that there is an intrinsic definition of a topological space, as a set plus some structure. Given that homotopy types are given to us as equivalence classes of topological spaces, one should ask whether a homotopy type could be described as a set plus some structure to be determined. The answer is no, one cannot: this is a theorem of Freyd from 1970. Formally, the homotopy category of spaces is not concretizable. So homotopy types are somehow "weird", from a traditional set-theoretic perspective.

[9] Cf. the Euler characteristic, for example.

[10] You might object: why just consider it as a *set*? Surely we can endow it with some topology, if X is reasonable (e.g. the compact-open topology), thus obtaining a simplicial *space* rather than a simplicial *set*. But at the end of the day you get the same information: the geometric realization of both these gadgets is homotopy equivalent to X. See MathOverflow/11025.

To be able to do this formally, we go to linear algebra[11] for aid.[12]

- The gadget $\mathrm{Sing}_\bullet(X)$ is not easy to manipulate, or rather, it is abstractly just as hard to manipulate as the space X itself.[13] But the added advantage is that it is very straightforwardly amenable to one of the most fruitful simplification processes in all of mathematics: linearization.

- We consider the gadget $\mathbb{Z}\mathrm{Sing}_\bullet(X)$, i.e. $\mathbb{Z}\mathrm{Sing}_n(X)$ is the free abelian group with basis the set of all maps $\Delta^n \to X$. This just means that we allow ourselves to make formal sums and differences of such maps.

  The n face maps $d_i : \mathbb{Z}\mathrm{Sing}_n(X) \to \mathbb{Z}\mathrm{Sing}_{n-1}(X)$ can be assembled, thanks to the device of taking "formal sums and differences", into a single map $d : \mathbb{Z}\mathrm{Sing}_n(X) \to \mathbb{Z}\mathrm{Sing}_{n-1}(X)$, $d = \sum_{i=0}^{n}(-1)^i d_i$ called the *boundary map*, which satisfies an essential property:

$$d^2 = 0.$$

  We get what is called a *chain complex*: a sequence of morphisms of abelian groups such that the composite of an arrow with the next one is zero.

$$\cdots \longrightarrow \mathbb{Z}\mathrm{Sing}_{n+1}(X) \xrightarrow{d} \mathbb{Z}\mathrm{Sing}_n(X) \xrightarrow{d} \mathbb{Z}\mathrm{Sing}_{n-1}(X) \longrightarrow \cdots$$

  This means that $\mathrm{Im}\left(\mathbb{Z}\mathrm{Sing}_{n+1}(X) \xrightarrow{d} \mathbb{Z}\mathrm{Sing}_n(X)\right) \subset \ker\left(\mathbb{Z}\mathrm{Sing}_n(X) \xrightarrow{d} \mathbb{Z}\mathrm{Sing}_{n-1}(X)\right)$.

  The difference between these two subgroups is the n-th *homology group* of X:

$$H_n(X) = \frac{\ker\left(\mathbb{Z}\mathrm{Sing}_n(X) \xrightarrow{d} \mathbb{Z}\mathrm{Sing}_{n-1}(X)\right)}{\mathrm{Im}\left(\mathbb{Z}\mathrm{Sing}_{n+1}(X) \xrightarrow{d} \mathbb{Z}\mathrm{Sing}_n(X)\right)}$$

  The elements of the numerator are called *cycles*, and those of the denominator are called *boundaries*.

- This definition is not as straightforward as that of homotopy, but with the algebra come extra tools for computation.[14] It is also a homotopy invariant: homotopy-equivalent spaces have isomorphic homologies.

- In our example of $U = \mathbb{R}^2 \setminus \{(0,0)\}$, the empty triangle around the origin is a cycle (for $n = 1$), but is not a boundary, hence defines a non-trivial element of $H_1(U)$. By contrast, $H_1(\mathbb{R}^2) = 0$, hence U and $\mathbb{R}^2$ can't be homotopy equivalent. [15]

[11] I should say abelian group theory and homological algebra, in all honesty.

[12] We could also go to vector calculus, as is perhaps familiar. Indeed, U admits a vector field $F(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$, whose line integral along the unit circle with the standard parametrization gives 2π. This can be used to prove that U is not diffeomorphic to $\mathbb{R}^2$: indeed, F is irrotational, but not a gradient. (This is secretly de Rham cohomology).

[13] What I'm thinking of here is that $\mathrm{Sing}_\bullet(X)$ is a *simplicial set*, and that for homotopy theory purposes, spaces and simplicial sets are equivalent through the Sing construction.

[14] For example, the long exact sequence and the Mayer-Vietoris sequence.

[15] Therefore the homology groups can also be said to detect "holes", albeit differently. For example, the "hole" in the torus is detected in homotopy by the π_1, but in homology by the H_2. Perhaps what is even more subtle than a "hole" is its "dimension" (neither of which have standard definitions). In any case, one could argue that a sensible definition of a "hole" is precisely a cycle which is not a boundary.

- Homology is a functor: for each $n \geq 0$ there is a functor H_n : Spaces $\to$ Abelian Groups. How much do these functors detect?

  A space X is *simply connected* if it is path-connected and $\pi_1(X, x) = 0$ for all $x \in X$.

  Theorem (Whitehead)**.** *Let X and Y be simply connected CW complexes. If a map $f : X \to Y$ induces an isomorphism in all of the H_n's, then f is a homotopy equivalence.*

  So, it detects the homotopy type *for simply connected spaces*. Homology is less powerful, but more computable.

- It turns out that there is also a version of homology for pointed spaces. This modification only affects the H_0, and is denoted $\tilde{H}_*$.

3 Homology through homotopy

- Since homology is less powerful than homotopy and homotopy is all-powerful, we might suspect that homology is a particular case of homotopy. What I mean is, from a based space X with basepoint $*$, can we find a new space $\hat{X}$ such that $\pi_*(\hat{X}) \cong \tilde{H}_*(X)$? [16]

- The answer is yes (if X is a CW complex). This was done by Dold and Thom.

  The abstract idea is the following. In homology, what we do is "linearize" each of the sets $\mathrm{Sing}_n(X)$. What we are now going to do is just make this "linearization" directly on the space, without passing through Sing. Hence our desired $\hat{X}$ will be the following $\mathbb{Z}^{\mathrm{top}}[X]$.

  The abstract definition of $\mathbb{Z}^{\mathrm{top}}[X]$ is: it is the *free topological abelian group* on X.[17] Explicitely, it is the quotient of

$$\left\{ \sum_i a_i x_i : a_i \in \mathbb{Z}, x_i \in X \right\} \tag{1}$$

  by the relations: $0x = *$ for all $x \in X$, $a* = *$ for all $a \in \mathbb{Z}$, and $nx + mx = (n+m)x$ for all $n, m \in \mathbb{Z}, x \in X$. Observe that this can alternately be described as

$$\left\{ \sum_i a_i x_i : a_i \in \mathbb{Z} \setminus \{0\}, x_i \in X \setminus \{*\}, x_i \neq x_j \text{ if } i \neq j \right\} \cup \{*\}. \tag{2}$$

  This can be imagined as: to each point $x \in X$ we can attach a weight $a \in \mathbb{Z}$. Now we consider all formal sums of points with weights, subject to the relations that adding the same point with two different weights amounts to considering the same point with the sum of the two weights, and the relations saying that the basepoint in X with any weight will serve as identity element.

  We have an abelian group structure, what about the topology? It is the quotient topology with respect to the surjection

$$\coprod_{n=0}^{\infty} \mathbb{Z}^n \times X^n \to \mathbb{Z}^{\mathrm{top}}[X], \qquad (a_1, \ldots, a_n, x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} a_i x_i$$

[16] Being picky, what we really want is a natural isomorphism of functors $\pi_* \circ \hat{-} \cong \tilde{H}_*$.

[17] To be interpreted as the free topological abelian group on a *pointed* space, where this means that we are asking that the basepoint serves as identity element. Also note that in Dold-Thom's original article this was denoted $AG(X)$.

where in the left hand side $\mathbb{Z}$ has the discrete topology.[18] Therefore $\mathbb{Z}^{\text{top}}[X]$ is a based space. The topology and the group structure are compatible.

- The theorem is that, for any CW complex X,

$$\tilde{H}_*(X) \cong \pi_*(\mathbb{Z}^{\text{top}}[X]).$$ [19]

  This explains the motto "homology is abelianized homotopy". The isomorphism is the following:

  Suppose $z = \sum a_i \sigma_i \in \mathbb{Z}\mathrm{Sing}_n(X)$. Then using the sum in $\mathbb{Z}^{\text{top}}[X]$, we can form a single map $\Delta^n \to \mathbb{Z}^{\text{top}}[X], t \mapsto \sum a_i \sigma_i(t)$.

  Now, if z is a cycle, then this maps descends to the quotient: $\Delta^n/\partial\Delta^n \to \mathbb{Z}^{\text{top}}[X]$. But $\Delta^n/\partial\Delta^n \cong S^n$, and we get a map $S^n \to \mathbb{Z}^{\text{top}}[X]$, whose homotopy class gives us the desired element in $\pi_n(\mathbb{Z}^{\text{top}}[X])$. [20]

- As an example, the previous theorem tells us that $\mathbb{Z}^{\text{top}}[S^n]$ is a model for an Eilenberg-Mac Lane space $K(\mathbb{Z}, n)$, i.e. the homotopy of $\mathbb{Z}^{\text{top}}[S^n]$ is concentrated in degree n.[21] This is because the homology of S^n is concentrated in degree n and of value $\mathbb{Z}$ (this fact would require a computation).

- Now, in some textbooks (such as Hatcher) you might see the Dold-Thom theorem in terms of a construction called the *infinite symmetric product*, SP, also present in Dold and Thom's original 1958 paper. Let us describe it, for completeness.

  First, consider the object $\mathbb{N}^{\text{top}}[X]$, defined as the quotient by the same relations of the set (1) where we have replaced the abelian group $\mathbb{Z}$ by the abelian monoid $\mathbb{N}$. It can also be described in a similar fashion to (2). Thus, it is the free abelian topological monoid on the based space X.

  This $\mathbb{N}^{\text{top}}[X]$ admits another description, which is the one that appears e.g. in Hatcher's textbook. Define $SP^n(X)$ to be the n-fold cartesian product X^n modulo the action of the symmetric group S_n on the factors, i.e. S_n acts on X^n as $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Define $SP(X)$ to be the colimit, i.e. the union $\bigcup_{n\geq 0} SP^n(X)$ under the inclusion maps $SP^n(X) \to SP^{n+1}(X)$ introducing the basepoint at the last coordinate. This is an alternative way of describing $\mathbb{N}^{\text{top}}[X]$ which does not work for the $\mathbb{Z}$-case (we don't have negative cartesian powers!).

- There is an obvious inclusion $\mathbb{N}^{\text{top}}[X] \to \mathbb{Z}^{\text{top}}[X]$. This induces isomorphisms in homotopy groups for X a connected CW complex. This is Satz 6.10.III in Dold-Thom's paper.[22] In particular, $\pi_*(SP(X)) \cong \tilde{H}_*(X)$. So SP also allows for interpreting homology through homotopy.

[18] Equivalently, the topology on $\mathbb{Z}^{\text{top}}[X]$ is the final topology with respect to the analogous maps $\mathbb{Z}^n \times X^n \to \mathbb{Z}^{\text{top}}[X]$, for all $n \geq 0$, i.e. the finest topology on $\mathbb{Z}^{\text{top}}[X]$ that makes these maps continuous.

[19] This can be taken as a *definition* of homology, and then the fact that we can express it via chain complexes, a "happy accident". This homotopical approach to homology is taken for example in the textbook of Aguilar, Gitler and Prieto.

[20] There is another nice proof, which is as follows. Show that $\pi_*(\mathbb{Z}^{\text{top}}[-])$ defines a reduced ordinary homology theory on based CW complexes, thus by uniqueness it coincides with $\tilde{H}_*$. This contains more information than the lone isomorphism above: it says the isomorphism is natural, as in the previous footnote, but it also says that the isomorphism commutes with the suspension isomorphisms. This is nice, because it implies that the long exact sequences they define are the same. Let me point out how they arise for $\pi_*(\mathbb{Z}^{\text{top}}[-])$: if $A \subset X$, then there is a fibration $\mathbb{Z}[A] \to \mathbb{Z}[X] \to \mathbb{Z}[X/A]$ (actually, a principal bundle) and the homology long exact sequence of $A \subset X$ is the homotopy long exact sequence of this fibration.

[21] More generally, if X is a Moore space of type (G, n), then $\mathbb{Z}^{\text{top}}[X]$ is an Eilenberg-Mac Lane space $K(G, n)$.

[22] For this to be true for large CW complexes, we need to work with a convenient category of spaces, e.g. compactly generated weakly Hausdorff.

- One can wonder whether these constructions could be extended to an arbitrary abelian monoid G, not only $\mathbb{N}$ or $\mathbb{Z}$, and even perhaps a topological one. This was done by McCord in 1969.[23]

4 Cohomology

- You may also have heard of the term "cohomology". (Ordinary) cohomology is a variant of homology obtained by dualization. Depending on the context, one may be more suitable than the other one. One main advantage of cohomology is that it is a better invariant, in the sense that it has more structure. It's not only a sequence of abelian groups, but actually a *graded ring*: there is a product. The existence of this product can help distinguish further spaces (i.e. there are spaces with isomorphic homology and cohomology groups, but different product in cohomology). [24]

5 (Extraordinary) (co)homology theories

- I don't have the time to discuss these. However, I want to stress that the homology we have discussed above is one very particular kind of "homology". One can axiomatize what one would desire of a *(co)homology theory*, by turning certain properties of ordinary (co)homology theory (as functors from spaces to abelian groups) into a definition, and then it turns out that there is a whole zoo of (co)homology theories fundamentally different from the ordinary one. Three big important ones are topological K-theory, cobordism, and stable homotopy. They have been fundamental in the resolution of several classical topological problems that predate algebraic topology.[25]

  One could alternatively take the Dold-Thom theorem as starting point and think that we will get different homology theories by turning certain properties of SP into a definition. Then taking homotopy groups would give us a new "generalized homology theory". This is a valid approach.[26]

  An important point is that these (co)homology theories do *not* come from a chain complex.[27]

- Another more classical way in which a homology theory can be obtained through homotopy is through the theory of *spectra*, but that's a story better left for another day.[28]

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[23] We get the nice result that for G an abelian topological monoid, $G^{\text{top}}[S^1]$ is a model for BG, the classifying space of G.

[24] One might wonder about this asymmetry. An observation for the initiated is that whereas the cohomology of a space is an algebra, the homology is a *coalgebra*... if one works with field coefficients. (The problem for integral coefficients is that the splitting in the Künneth short exact sequence is not natural!)

[25] E.g. the only spheres that have a trivial tangent bundle are S^0, S^1, S^3 and S^7, first proven by Adams using K-theory.

[26] The hypotheses on the functors are: "pointed, 1-excisive homotopy functors", the most non-trivial thing perhaps being that cofibration sequences get sent to fibration sequences. See e.g. section 1 of Goodwillie's Calculus I paper. Also, see MathOverflow/182823 for a construction due to Segal of connective K-homology in this spirit.

[27] In fact, the only one coming from a chain complex is the singular one (and sensible amalgamations of these, amounting to wedges of Eilenberg-Mac Lane spectra). This is a theorem of Burdick, Conner and Floyd.

[28] For the initiated: for a spectrum E, we have its associated homology theory $E_*(X) = \pi_*(\Omega^\infty(E \wedge X))$. We see the link to the Dold-Thom approach. The space $\Omega^\infty(E \wedge X)$ is not very explicit, since it involves the Ω-spectrum replacement of $E \wedge X$. However, the paper *Partial monoids and Dold-Thom functors* by Mostovoy generalizes the Dold-Thom construction from the Eilenberg-Mac Lane spectrum giving ordinary homology, to general spectra.