A construction of the K-theory spectrum of a ring

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Abstract

The purpose of this note is to sketch a construction of the K-theory Ω-spectrum of a ring. Virtually all the results of all but the last paragraph are stated with this construction in view. We assume the reader has some familiarity with homotopy theory but we don't assume any familiarity with K-theory.

Throughout this note, R will denote a ring (associative and unital). All CW-complexes are taken to be pointed; a *space* means a (pointed) CW-complex, and a *map of spaces* means a continuous based map between spaces.

§1 Introduction During the 1950s and the 1960s, three abelian groups were constructed from a ring R:

- $K_0(R)$, the group completion of the commutative monoid $\mathrm{Proj}(R)$ of finitely generated projective R-modules under direct sum.
- $K_1(R)$, the abelianization of the group $GL(R)$, i.e. the quotient $GL(R)/[GL(R), GL(R)]$. Here $GL(R)$ is the group arising as the direct limit of the inclusions $GL_n(R) \hookrightarrow GL_{n+1}(R)$, $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $[GL(R), GL(R)]$ denotes the commutator subgroup of $GL(R)$. It can be proven ("Whitehead's lemma") that $[GL(R), GL(R)] = E(R)$ and thus $K_1(R) = GL(R)/E(R)$. Here $E(R)$ denotes the group arising as the direct limit (under inclusions as above) of the groups $E_n(R)$ generated by the elementary matrices. An *elementary matrix* $A \in E_n(R)$ is an $n \times n$ matrix that has 1 in every diagonal spot, an element $r \in R$ in one non-diagonal entry (i, j), and is zero elsewhere. We will denote such a matrix by e_{ij}^r.
- $K_2(R)$ has a more complicated definition.

They were thus numbered for several reasons. For example, we have

$$K_1(R) \cong H_1(GL(R); \mathbb{Z}) \quad \text{and} \quad K_2(R) \cong H_2(E(R); \mathbb{Z}),$$

where $H_i(-; \mathbb{Z})$ denotes integral group homology.

Another reason is the following. There are also "relative K-groups" of a pair (R, I) where I is an ideal of R; these are denoted $K_i(R, I)$ for $i = 1, 2$. These groups fit into a long exact sequence:[1]

$$K_2(R, I) \to K_2(R) \to K_2(R/I) \to K_1(R, I) \to K_1(R) \to K_1(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I).$$

Quillen was the one to give, in 1973, the "right" generalization of the K_i-groups to higher i. His strategy was to attach a space KR to the ring R, in a way that its homotopy groups would be the K-groups of R. The exact sequence above continues to the left, and arises as the long exact homotopy sequence of a fibration.

The space KR was proven to be an infinite loop space[2], thus fitting into the 0-th stage of an Ω-spectrum. In this note we will outline one of several ways to construct this object.

§2 Perfect and quasi-perfect groups

Definition. Let G be a group. We say it is *perfect* if it equals its commutator, i.e. $G = [G, G]$. Equivalentely, its abelianization G_{ab} is trivial.

We say it is *quasi-perfect* if its commutator is perfect.

Of course, every perfect group is quasi-perfect.

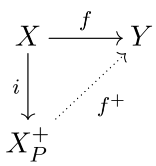
Proposition ([6, (2.1.4)]). *$GL(R)$ is quasi-perfect, i.e. $E(R)$ is perfect.*

§3 The "+"-construction There is a universal way to modify a connected space so as to kill a chosen perfect normal subgroup of its fundamental group, without altering its homology. This is the "+"-construction which we state in the following

Theorem ([1, (3.1, 3.2)]). *Let X be a connected CW-complex and P a perfect normal subgroup of $\pi_1(X)$.*

There exists a connected CW-complex X_P^+, obtained from X by attaching 2-cells and 3-cells, such that the inclusion $i : X \hookrightarrow X_P^+$ satisfies the following properties:

1. *$\pi_1(X_P^+) = \pi_1(X)/P$ and the induced homomorphism $i_* : \pi_1(X) \to \pi_1(X_P^+)$ is the quotient map $\pi_1(X) \to \pi_1(X)/P$,*
2. *i induces an isomorphism $i_* : H_*(X; L) \to H_*(X_P^+; L)$ for any local coefficient system L on X_P^+,*
3. *the pair (X_P^+, i) satisfies the following universal property: if Y is a connected CW-complex and $f : X \to Y$ is a map such that $f_* : \pi_1(X) \to \pi_1(Y)$ satisfies $f_*(P) = 0$, then there exists a unique map $f^+ : X_P^+ \to Y$ up to homotopy making the following diagram homotopy commute.*



The "+"-construction is thus functorial up to homotopy.

[1] The group $K_0(I)$ can be conceived as the K-group of a ring without unit [2, (§2.1)], or through the notion of the "augmented ring" $R \oplus I$ [10, (II.2.3), p. 242].

[2] With our definitions it will be, more precisely, the space $K_0(R) \times KR$ where $K_0(R)$ is given the discrete topology.

§4 The classifying space of a group

Theorem ([2, (6.1)]). *Let G be a group. There exists a (connected) CW-complex BG such that $\pi_1(BG) = G$ and $\pi_i(BG) = 0$ for $i \neq 1$. Such a space is called a* classifying space for G *or a* $K(G,1)$ space, *and is unique up to homotopy equivalence.*

There exists a functorial construction for BG: we get a functor B from the category of groups to the category of CW-complexes.

If G is a quasi-perfect group with commutator N, we may apply the "+"-construction to the space BG with respect to the perfect normal subgroup $N \subset G = \pi_1(BG)$. In this case, we will denote BG_N^+ by BG^+.

Proposition ([2, (6.2.1.ii)]). *If G is a quasi-perfect group and N is its commutator, then the map $BN^+ \to BG^+$ induced by the inclusion $N \hookrightarrow G$ is a universal covering map.*

§5 The K-space of R Since $GL(R)$ is quasi-perfect with commutator $E(R)$, we can form the space $BGL(R)^+ = BGL(R)^+_{E(R)}$. This space satisfies $\pi_1(BGL(R)^+) = GL(R)/E(R) = K_1(R)$. It can be proven that $\pi_2(BGL(R)^+) = K_2(R)$ ([10, (IV.1.7.1)]). This leads us to make the following

Definition. Let R be a ring. We define the K*-space* of R as the topological space

$$K(R) := BGL(R)^+.$$

We define the K_i*-groups* of R for $i \geq 1$ to be the homotopy groups of $K(R)$:

$$K_i(R) := \pi_i K(R) \qquad \text{for } i \geq 1.$$

We observed above that this definition coincides with the classical one for $i = 1, 2$.

The assignment $R \mapsto K(R)$ can be taken to be (strictly) functorial, see [2, (6.3)] for details. It can also be extended functorially to non-unital rings, see [2, p. 28].

We might have defined $K(R)$ as $K_0(R) \times BGL(R)^+ = \bigsqcup_{K_0(R)} BGL(R)^+$, where $K_0(R)$ is given the discrete topology. The homotopy groups of this space coincide with the ones for $BGL(R)^+$ for $i \geq 1$, and we get the advantage that if we apply π_0 we recover the set $K_0(R)$ since $BGL(R)^+$ is connected.

However, this is not right in the categorical sense: "the problem is that one cannot write $K(R)$ functorially as a product of $K_0(R)$ and $BGL(R)^+$", remarks Schlichting in [7, (2.2.9)].

§6 Cone and suspension

Definition. The *cone* of R is the ring CR of row-and-column-finite matrices over R, i.e. the ring of infinite matrices that have finite non-zero elements in every row and column.

Denote by MR the set of infinite matrices over R with finite non-zero coefficients, i.e. the colimit of the inclusions $M_n(R) \hookrightarrow M_{n+1}(R)$, $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

The set $MR \subset CR$ is an ideal; the quotient ring $\Sigma R := CR/MR$ is called the *suspension* of R.

Proposition. • ([9, (2.5) and below], [4, (1.4.7)]) *The space $K(CR)$ is contractible and CR satisfies $K_i(CR) = 0$ for all $i \geq 0$.*

• ([2, p. 8 and (6.2.6.iii)]) *There is a homotopy equivalence $K(MR) \simeq K(R)$ and MR satisfies $K_i(MR) = K_i(R)$ for all $i \geq 0$.*

The previous proposition and the long exact sequence in §1 yield the following

Proposition. $K_1(\Sigma R) \cong K_0(R)$.

§7 The K-theory Ω-spectrum If X is a space, denote by $\Omega_0 X \subset \Omega X$ the connected component of the trivial loop.

Lemma. *Let X be a space and let $p : \widetilde{X} \to X$ be a universal cover. There is a homotopy equivalence $\Omega\widetilde{X} \to \Omega_0 X$.*

Proof. Consider the map $\Omega p : \Omega\widetilde{X} \to \Omega X$. The space $\Omega\widetilde{X}$ is connected, since $\pi_0(\Omega\widetilde{X}) \cong \pi_1(\widetilde{X}) = 0$. Therefore the image of Ωp lies in $\Omega_0 X$: we have a map

$$\Omega p : \Omega\widetilde{X} \to \Omega_0 X. \tag{1}$$

Let us check this map is a weak equivalence. Firstly, both its domain and codomain are connected. Secondly, since the homotopy groups for $i \geq 1$ only depend on the connected component of the base point, we have the equality in the following commutative diagram.

$$\begin{array}{ccc} \pi_i\Omega\widetilde{X} & \xrightarrow{(\Omega p)_*} & \pi_i\Omega_0 X = \pi_i \Omega X \\ \cong\downarrow & & \downarrow\cong \\ \pi_{i+1}\widetilde{X} & \xrightarrow[p_*]{} & \pi_{i+1}X \end{array}$$

The map $p_* : \pi_{i+1}\widetilde{X} \to \pi_{i+1}X$ is an isomorphism since $i + 1 \geq 2$. Therefore the map $(\Omega p)_* : \pi_i\Omega\widetilde{X} \to \pi_i\Omega_0 X$ is one too.

Since loop spaces of CW-complexes have the homotopy type of a CW-complex, the map (1) is a homotopy equivalence by Whitehead's theorem. □

Proposition ([9, (3.1)]). *Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of groups. Suppose (i) G_1 and G_2 are quasi-perfect and G_3 is perfect, (ii) for every $g \in G_1, h_1, \ldots, h_n \in G_2$ there exists $h \in G_2$ such that $gh_jg^{-1} = hh_jh^{-1}$. Then there is a homotopy fibration sequence*

$$BG_1^+ \longrightarrow BG_2^+ \longrightarrow BG_3^+ .$$

Theorem. *We have a homotopy equivalence*

$$K(R) \xrightarrow{\simeq} \Omega_0 K(\Sigma R) . \tag{2}$$

Proof. Consider the following short exact sequence in **Rng**.

$$0 \longrightarrow MR \xrightarrow{\ i\ } CR \xrightarrow{\ p\ } \Sigma R \longrightarrow 0$$

The functor GL : **Ring** $\to$ **Grp** extends to a left exact functor **Rng** $\to$ **Grp** [2, (2.1.10)]. We thus have an exact sequence in **Grp**:

$$1 \longrightarrow GL(MR) \xrightarrow{GLi} GL(CR) \xrightarrow{GLp} GL(\Sigma R)\ .$$

On the other hand, the induced map $E(CR) \to E(\Sigma R)$ is surjective. Indeed, the functor E : **Ring** $\to$ **Grp** preserves surjections, since it acts like $(Ep)(e_{ij}^r) = e_{ij}^{p(r)}$. By a proposition in §6 we have $K_1(CR) = 0$, hence $GL(CR) = E(CR)$. We have the following commutative square.

$$\begin{array}{ccc} GL(CR) & \xrightarrow{GLp} & GL(\Sigma R) \\ {\scriptstyle =}\uparrow & & \uparrow \\ E(CR) & \xrightarrow[Ep]{} & E(\Sigma R) \end{array}$$

This shows that $\ker Ep = \ker GLp$, and we thus get a short exact sequence in **Grp**:

$$1 \longrightarrow GL(MR) \longrightarrow GL(CR) \longrightarrow E(\Sigma R) \longrightarrow 1\ .$$

We have observed in §2 that the groups in this short exact sequence satisfy the first condition of the proposition above. The second condition is also satisfied ([9, p. 357]). Since $K(MR) \simeq K(R)$ (§6), we obtain a homotopy fibration sequence

$$K(R) \longrightarrow K(CR) \longrightarrow BE(\Sigma R)^+\ .$$

Since $K(CR)$ is contractible (§6), the homotopy fiber of the second map is $\Omega BE(\Sigma R)^+$, and we get a homotopy equivalence

$$\Omega BE(\Sigma R)^+ \xrightarrow{\ \simeq\ } K(R). \tag{3}$$

The proposition in §4 gives that $BE(\Sigma R)^+ \to K(\Sigma R)$ is a universal cover. The previous lemma yields a homotopy equivalence

$$\Omega BE(\Sigma R)^+ \xrightarrow{\ \simeq\ } \Omega_0 K(\Sigma R)\ . \tag{4}$$

Composing (4) with a homotopy inverse for (3) gives the desired homotopy equivalence. □

Corollary. *If* $n \geq 1$, *then* $K_n(\Sigma R) \cong K_{n-1}(R)$.

Proof. For $n = 1$ this was observed in §6.

If $n \geq 2$, since the homotopy groups of a space depend only on the connected component of the base point for $n-1 \geq 1$, we have

$$K_n(\Sigma R) = \pi_n\big(K(\Sigma R)\big) = \pi_{n-1}\big(\Omega K(\Sigma R)\big) = \pi_{n-1}\big(\Omega_0 K(\Sigma R)\big) \cong \pi_{n-1}(KR) = K_{n-1}(R).$$

□

Definition. We define the *K-theory* Ω-spectrum of R, denoted $\mathbb{K}R$, as follows. Define

$$(\mathbb{K}R)_n := \Omega K(\Sigma^{n+1}R) \qquad \text{for } n \geq 0.$$

The structure maps are given by the homotopy equivalences obtained from (2) applied to the ring $\Sigma^{n+1}R$, again using that loop spaces depend only on the connected component of the base point:

$$(\mathbb{K}R)_n = \Omega K(\Sigma^{n+1}R) \xrightarrow{\simeq} \Omega\Omega_0 K(\Sigma^{n+2}R) = \Omega^2 K(\Sigma^{n+2}R) = \Omega(\mathbb{K}R)_{n+1} .$$

Observe that $(\mathbb{K}R)_0$ is homotopy equivalent to $K_0(R) \times K(R)$ where $K_0(R)$ is given the discrete topology. Indeed, $(\mathbb{K}R)_0 = \Omega K(\Sigma R)$: on one hand the path component of its base point is $\Omega_0 K(\Sigma R) \simeq K(R)$, and on the other hand, we have

$$\pi_0(\Omega K(\Sigma R)) \cong \pi_1(K(\Sigma R)) = K_1(\Sigma R) \cong K_0(R).$$

Since the path components of a loop space are all homotopy equivalent [3], we have a homotopy equivalence $(\mathbb{K}R)_0 \simeq K_0(R) \times K(R)$.

The spectrum $\mathbb{K}R$ we have just defined is generally nonconnective, i.e. its negative homotopy groups might be non-trivial. Its non-negative homotopy groups yield the K-groups:

Proposition. *Let $n \geq 0$. Then $\pi_n(\mathbb{K}R) = K_n(R)$.*

Proof.

$$\begin{aligned}
\pi_n(\mathbb{K}R) &= \operatorname*{colim}_p \pi_{n+p}((\mathbb{K}R)_p) \\
&= \operatorname*{colim}_p \pi_{n+p}\big(\Omega K(\Sigma^{p+1}R)\big) \\
&= \operatorname*{colim}_p \pi_{n+p+1}\big(K(\Sigma^{p+1}R)\big) \\
&= \operatorname*{colim}_p K_{n+p+1}(\Sigma^{p+1}R) \\
&= \operatorname*{colim}_p K_n(R) \\
&= K_n(R)
\end{aligned}$$

□

The negative homotopy groups of $\mathbb{K}R$ can be proven to coincide with the negative K-groups of R introduced by Bass [10, (IV.10.4.1)].

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