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An algebraic description of the algebraic K-theory groups

Author:
Bruno Stonek

Supervisor:
Prof. Dr. Christian Ausoni

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Abstract

In this dissertation we give a description through generators and relations of the K-groups of an exact category $\mathcal{N}$ that supports long exact sequences. To this end, we define for every $n \geq 1$ a category $\Omega^n\mathcal{N}$ for which there are natural isomorphisms $K_n\mathcal{N} \cong K_0\Omega^n\mathcal{N}$, and we obtain a group presentation for the latter group. Throughout we use the theory of Waldhausen categories and their K-theory as introduced in [14]: we briefly review the concepts and theorems we will be using in the first chapter.

We claim no original content: all the main results are taken from [4].

Contents

Introduction

Higher algebraic K-theory may be said to have started with Quillen in his 1973 article [8]. He defined the notion of an *exact category* and a categorical construction, called the *Q-construction*, that assigns to an exact category $\mathcal{N}$ a category $Q\mathcal{N}$ in a functorial way. One may take the classifying space $BQ\mathcal{N}$ of this category: the loop space of $BQ\mathcal{N}$ is the *K-space of $\mathcal{N}$*, noted $K\mathcal{N}$ and proven to be an infinite loop space, giving rise to an Ω-spectrum. Its homotopy groups give the higher K-groups. Quillen proved his definition to coincide with the classical lower K-groups of a ring R, when applying it to the category $\mathrm{Proj}(R)$ of finitely generated projective modules over R. His development of this theory convinced the mathematical community that his generalization was a fruitful one.

Quillen was the first to define the K-groups as the homotopy groups of a certain space. In the following years new approaches and generalizations appeared. For example, Gillet and Grayson [3] attached a simplicial set $G\mathcal{N}$ to the exact category $\mathcal{N}$, whose i-th homotopy group yielded the K_i-group of Quillen, without having to loop. Waldhausen [14] extended the scope of K-theory to *categories with cofibrations and weak equivalences*, more general than exact categories and nowadays called *Waldhausen categories*, through a construction called the *S-construction*. He proved both coincided in the exact case.

The K_0 group of an exact category admits an explicit presentation which is a direct generalization of the classical definition for rings. In 1996, Nenashev [7] found a presentation for the K_1 group of an exact category, through *double short exact sequences*, using the methods of Gillet and Grayson.

In 2012, Grayson [4], inspired by this result of Nenashev and using the machinery of Waldhausen, found a presentation for the higher K-groups of an exact category, through a generalization of double short exact sequences, called *acyclic binary multicomplexes*. This is the first time a completely algebraic description of the higher K-theory groups of Quillen has been found. Morally, the strategy is to attach, to the exact category $\mathcal{N}$, an exact category $\Omega^n\mathcal{N}$, such that $K\mathcal{N}$ is an n-delooping of $K\Omega^n\mathcal{N}$. This is the content of theorem 3.3.3. A presentation for $K_n\mathcal{N}$ is then found through a presentation of the π_0-group of $K\Omega^n\mathcal{N}$.

We will assume the reader has some familiarity with abelian categories and with homotopy theory. We do not assume any knowledge of exact categories, Waldhausen categories or algebraic K-theory.

Chapter 1

Setting the stage

1.1 Exact categories

In this section we will fix some terminology on exact categories. We take our main definition from [1, (2.1)].

Definition 1.1.1. Let $\mathcal{N}$ be an additive category. A *kernel-cokernel pair* (i, p) is a sequence $N' \xrightarrow{i} N \xrightarrow{p} N''$ in $\mathcal{N}$ such that $i = \ker p$ and $p = \operatorname{coker} i$.

Let us fix a class $\mathcal{E}$ of kernel-cokernel pairs in $\mathcal{N}$. We say that a morphism i (resp. p) is an *admissible monomorphism* (resp. *admissible epimorphism*) if there exists a morphism p (resp. i) such that $(i, p) \in \mathcal{E}$.

An *exact structure* in $\mathcal{N}$ is a class $\mathcal{E}$ of kernel-cokernel pairs that satisfies the following conditions:

- $\mathcal{E}$ is closed under isomorphism,
- for every $N \in \mathcal{N}$ the identity morphism id_N is an admissible monomorphism and an admissible epimorphism,
- the class of admissible monomorphisms and the class of admissible epimorphisms are closed under composition,
- The pushout (resp. pullback) of an admissible monomorphism (resp. epimorphism) along an arbitrary morphism exists and is an admissible monomorphism (resp. epimorphism).

An *exact category* is a pair $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ is an additive category and $\mathcal{E}$ is an exact structure on $\mathcal{N}$. The elements (i, p) of $\mathcal{E}$ are called *short exact sequences in* $\mathcal{N}$ and are denoted as

$$0 \longrightarrow N' \xrightarrow{i} N \xrightarrow{p} N'' \longrightarrow 0 \tag{1.1}$$

We will usually say that $\mathcal{N}$ (instead of $(\mathcal{N}, \mathcal{E})$) is an exact category if there is no danger of confusion.

Convention 1.1.2. In order to avoid size issues, we will make the convention that exact categories be skeletally small.

A functor $\mathcal{N} \to \mathcal{N}'$ between exact categories is called *exact* if it is additive and it carries short exact sequences in $\mathcal{N}$ to short exact sequences in $\mathcal{N}'$. We say it *reflects exactness* if exactness of the image in $\mathcal{N}'$ of a sequence of maps $N_1 \to N_2 \to N_3$ in $\mathcal{N}$ forces exactness of the given sequence.

We thus have a category **Exact** whose objects are exact categories and whose arrows are exact functors.

Any abelian category has a natural exact structure where "short exact sequence" has its usual meaning, and every monomorphism and epimorphism is admissible.

General exact categories are characterized by the fact that they are full subcategories of abelian categories which are closed under extension:

Theorem 1.1.3. *Let $\mathcal{N}$ be an exact category. There exists an abelian category $\mathcal{A}$ and a fully faithful exact functor $i : \mathcal{N} \to \mathcal{A}$ that reflects exactness and such that $\mathcal{N}$ is closed under extensions in $\mathcal{A}$, i.e. if (1.1) is a short exact sequence in $\mathcal{A}$ with N', N'' in $i(\mathcal{N})$, then N is isomorphic to an object of $i(\mathcal{N})$.*

Thus the short exact sequences of $\mathcal{N}$ are identified with the short exact sequences of $\mathcal{A}$ whose objects are in $i(\mathcal{N})$.

We say that $i : \mathcal{N} \hookrightarrow \mathcal{A}$ is an *admissible embedding* of $\mathcal{N}$.

For a proof, see [12, (A.7.1)] or [1, (A.1)]. We can see in [16, (Ex. II.7.8)] that if we define an exact category as an additive category having an admissible embedding, then it is an exact category by our definition.

Example 1.1.4. Let R be a ring and $\mathrm{Proj}(R)$ be the category of finitely generated projective modules over R. It is an exact category, as it is a full subcategory of the category of R-modules which is closed under extensions.

It is not an abelian category in general, though, since for instance it does not have all cokernels. For example, the morphism $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ in $\mathrm{Proj}(\mathbb{Z})$ has $\mathbb{Z}/2\mathbb{Z}$ as cokernel, which is not projective, since projective $\mathbb{Z}$-modules are free, but $\mathbb{Z}/2\mathbb{Z}$ is torsion.

1.2 Waldhausen categories

We take our definitions from Waldhausen's foundational article [14].

Definition 1.2.1. A *category with cofibrations* is a category with a fixed zero object $*$ endowed with a subcategory $\mathrm{co}\, \subset \mathcal{C}$ whose arrows, denoted by $A \rightarrowtail B$, are called *cofibrations*. They satisfy:

1. the isomorphisms of $\mathcal{C}$ are cofibrations,
2. for every $A \in \mathcal{C}$ the unique arrow $* \to A$ is a cofibration,
3. the pushout of a cofibration along an arbitrary morphism exists and is a cofibration.

We remark that conditions 1 or 2 imply that co is a *lluf subcategory*, i.e. every object of $\mathcal{C}$ is in co.

If $A \rightarrowtail B$ is a cofibration, we define B/A as a pushout: $:= * \cup_A B$, and we denote by $B \twoheadrightarrow B/A$ the canonical map, called the *quotient map*. The sequence $A \rightarrowtail B \twoheadrightarrow B/A$ is called a *cofibration sequence* in $\mathcal{C}$.

Observe that in a category with cofibrations we also have finite coproducts: if $A, B \in \mathcal{C}$, then $A \cup_* B$ is a coproduct of A and B in $\mathcal{C}$, which we will denote by $A \sqcup B$.

Definition 1.2.2. A *Waldhausen category* is a category with cofibrations $\mathcal{C}$ endowed with a subcategory $w \subset \mathcal{C}$ whose arrows, denoted by $A \xrightarrow{\simeq} B$, are called *weak equivalences*. They satisfy:

1. the isomorphisms of $\mathcal{C}$ are in w,
2. (Gluing axiom). In the following commutative diagram, where $A \rightarrowtail B$ and $A' \rightarrowtail B'$ are cofibrations, if the three solid vertical arrows are weak equivalences, then the induced dashed arrow is one too.

$$\begin{tikzcd}
 & B \arrow[rr] \arrow[dd, "\simeq"] & & B \cup_A C \arrow[dd, dotted, "\simeq"] \\
A \arrow[ur, tail] \arrow[rr] \arrow[dd, "\simeq"'] & & C \arrow[ur, tail] \arrow[dd, "\simeq"] & \\
 & B' \arrow[rr] & & B' \cup_{A'} C' \\
A' \arrow[ur, tail] \arrow[rr] & & C' \arrow[ur, tail] &
\end{tikzcd}$$

A Waldhausen category is thus a triple $(\mathcal{C}, \mathrm{co}\,, w)$ but we will always denote one such triple by $w\mathcal{C}$ or by $\mathcal{C}$ if we need not specify the subcategory of weak equivalences.

Observe that the gluing axiom implies that subcategories of weak equivalences are closed under coproducts.

Convention 1.2.3. In order to avoid size issues, we will make the convention that Waldhausen categories be skeletally small.

An *exact functor* $F : \mathcal{C} \to \mathcal{D}$ between Waldhausen categories is a functor that preserves all the relevant structure: zero object, cofibrations, weak equivalences and pushouts along cofibrations. This defines a category **Wald** with Waldhausen categories as objects and exact functors as arrows.

Example 1.2.4. Any exact category is a Waldhausen category. The admissible monomorphisms are the cofibrations and the isomorphisms the weak equivalences. The gluing condition follows from functoriality of pushouts.

In our particular applications we will consider exact categories with these cofibrations but we will consider different categories of weak equivalences.

There are some additional axioms a Waldhausen category $\mathcal{C}$ may satisfy that can be handy. These are the following:

- (Saturation axiom). If f and g are composable arrows in $\mathcal{C}$ and two out of three of f, g or fg are weak equivalences, then so is the third.

- (Extension axiom). Suppose we have the following commutative diagram where the rows are cofibration sequences:

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d] & B/A \arrow[d] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & B/A
\end{tikzcd}$$

  If the left and right vertical maps are weak equivalences, then so is the middle one.

Definition 1.2.5. Let J be the following diagram category:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[dr] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[dl] \\
 & \bullet &
\end{tikzcd}$$

A *cylinder functor* in a Waldhausen category $\mathcal{C}$ is a functor $\operatorname{Arr}\mathcal{C} \to \mathcal{C}^J$ that carries an arrow $f : A \to B$ in $\mathcal{C}$ to a diagram like the following

$$\begin{tikzcd}
A \arrow[r, "j_1"] \arrow[dr, "f"'] & \mathbb{T}(f) \arrow[d, "p"] & B \arrow[l, "j_2"'] \arrow[dl, "\mathrm{id}_B"] \\
 & B &
\end{tikzcd} \tag{1.2}$$

and it satisfies the following for every arrow $f : A \to B$ in $\mathcal{C}$ and every arrow $f \to f'$ in $\operatorname{Arr}\mathcal{C}$ as depicted in the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

(i) the diagram (1.2) commutes,

(ii) $\mathbb{T}(* \to A) = A$ and in this case $j_2 = p = \mathrm{id}_A$,

(iii) the map $A \sqcup B \xrightarrow{j_1 \sqcup j_2} \mathbb{T}(f)$ is a cofibration,

(iv) if α and β are weak equivalences, then so is the induced map $\mathbb{T}(f) \to \mathbb{T}(f')$,

(v) if α and β are cofibrations, then so are the induced map $\mathbb{T}(f) \to \mathbb{T}(f')$ and the induced dashed arrow in the following pushout diagram:

$$\begin{tikzcd}
A \sqcup B \arrow[r, tail, "j_1 \sqcup j_2"] \arrow[d, tail, "\alpha \sqcup \beta"'] & \mathbb{T}(f) \arrow[d, tail] \arrow[ddr, bend left] & \\
A' \sqcup B' \arrow[r, tail] \arrow[drr, tail, bend right, "j_1' \sqcup j_2'"'] & P \arrow[dr, dotted] & \\
 & & \mathbb{T}(f')
\end{tikzcd}$$

We call j_1 the *front inclusion*, j_2 the *back inclusion* and p the *projection* of $\mathbb{T}(f)$.

The following axiom may be satisfied by a Waldhausen category $\mathcal{C}$ with a cylinder functor:

- (Cylinder axiom). For every $f : A \to B$ in $\mathcal{C}$, the projection $p : \mathbb{T}(f) \to B$ is a weak equivalence.

Remark 1.2.6. Let $w\mathcal{C}$ be Waldhausen category with a cylinder functor, satisfying the saturation and cylinder axioms. If $f : A \to B$ is a weak equivalence, then so is the front inclusion j_1, because $f = p \circ j_1$.

The following proposition is discussed in [12, (1.8.1)].

Proposition 1.2.7. *Let $w\mathcal{C}$ be a Waldhausen category. Denote by $\mathcal{C}^w$ the full subcategory of $\mathcal{C}$ whose objects $A \in \mathcal{C}$ are the ones such that the unique arrow $* \to A$ is a weak equivalence (the "w-acyclic" objects of $\mathcal{C}$). Then $w\mathcal{C}^w$ is a Waldhausen category, where we understand that the w on the left means $w \cap \mathcal{C}^w$.*

If $\mathcal{C}$ is in fact an exact category and $w \subset \mathcal{C}$ satisfies the extension axiom, then $\mathcal{C}^w$ is an exact category.

Remark 1.2.8. If $A \in \mathcal{C}^w$, $B \in \mathcal{C}$ and there is a weak equivalence $A \to B$, then $B \in \mathcal{C}^w$ since w is closed under composition.

1.3 Chain complexes

In this section we fix terminology on chain complexes. Let $\mathcal{N}$ be an exact category.

We will denote by $\mathrm{Gr}\mathcal{N}$ the category of bounded $\mathbb{Z}$-graded objects in $\mathcal{N}$ and by $C\mathcal{N}$ the category of bounded chain complexes in $\mathcal{N}$. We will denote the objects of $C\mathcal{N}$ as (N, d) where $N \in \mathrm{Gr}N$ and d is a differential on N. If there is no danger of confusion, we will omit the differential from the notation. In this case, if we say that N is a chain complex, we will denote by $\mathrm{gr}N$ its underlying graded object.

The categories $\mathrm{Gr}\mathcal{N}$ and $C\mathcal{N}$ are exact categories. Indeed, if $\mathcal{N} \hookrightarrow \mathcal{A}$ is an admissible embedding, then $\mathrm{Gr}\mathcal{A}$ and $C\mathcal{A}$ are abelian categories and the induced functors $\mathrm{Gr}\mathcal{N} \to \mathrm{Gr}\mathcal{A}$ and $C\mathcal{N} \to C\mathcal{A}$ are admissible embeddings.

If $N \in \mathrm{Gr}\mathcal{N}$ and $n \in \mathbb{Z}$, we denote by $N[n]$ the bounded graded object that has N_{i+n} in degree i. If $N = (\mathrm{gr}N, d) \in C\mathcal{N}$, we denote by $N[n]$ the bounded chain complex that has $\mathrm{gr}N[n]$ as underlying graded object and $(-1)^n d$ as differential.

If $n > m$ are integers, we say that a bounded graded object N or bounded chain complex (N, d) is *supported on the interval* $[n, m]$ if $N_i = 0$ for all $i > n$ and $i < m$, and we say it has *length* $n - m$.

Seeing as we will only be working with the bounded case, we make the following

Convention 1.3.1. We will make the convention that graded objects and chain complexes be bounded.

These constructions obviously define functors $\mathrm{Gr}, C : \mathbf{Exact} \to \mathbf{Exact}$. If $n \in \mathbb{Z}$, shifting defines natural transformations

$$\begin{tikzcd} \mathbf{Exact} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, bend right, "\mathrm{Gr}"'] \arrow[r, phantom, "\Downarrow{[n]}"] & \mathbf{Exact} \end{tikzcd} \text{ and } \begin{tikzcd} \mathbf{Exact} \arrow[r, bend left, "C"] \arrow[r, bend right, "C"'] \arrow[r, phantom, "\Downarrow{[n]}"] & \mathbf{Exact} \end{tikzcd}.$$

1.3.1 Acyclicity

Recall that in an abelian category, a chain complex is exact if and only if it is obtained by splicing together short exact sequences. Since in a general exact category we don't have kernels or cokernels, we can't make the usual definition of exactness, but seeing as we have short exact sequences, we promote this splicing criterion to a definition.

Definition 1.3.2. Let $(N, d) \in C\mathcal{N}$. We say (N, d) is *acyclic* or that it is a *long exact sequence* if it is obtained by splicing together short exact sequences of $\mathcal{N}$ (called its *component short exact sequences*). More explicitely, for every $i \in \mathbb{Z}$ there exist $Z_i \in \mathcal{N}$ and arrows $p_i : N_i \to Z_{i-1}$, $j_i : Z_i \to N_i$ such that $d_i : N_i \to N_{i-1}$ can be factored as $N_i \xrightarrow{p_i} Z_{i-1} \xrightarrow{j_{i-1}} N_{i-1}$ and

$$0 \longrightarrow Z_i \xrightarrow{j_i} N_i \xrightarrow{p_i} Z_{i-1} \longrightarrow 0$$

is a short exact sequence in $\mathcal{N}$. We display this data in the following commutative diagram where the "$\wedge$"-shaped sequences are short exact:

$$\begin{tikzcd}
\cdots \arrow[rr] & & N_i \arrow[rr, "d_i"] \arrow[dr, "p_i"] & & N_{i-1} \arrow[rr, "d_{i-1}"] \arrow[dr, "p_{i-1}"] & & N_{i-2} \arrow[r] & \cdots \\
& Z_i \arrow[ur, "j_i"'] & & Z_{i-1} \arrow[ur, "j_{i-1}"'] \arrow[dr] & & Z_{i-2} \arrow[ur, "j_{i-2}"'] \arrow[dr] & & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

Remark 1.3.3.

1. The differentials d_i in an acyclic chain complex have a kernel and a cokernel, and they are j_i and p_{i-1} respectively. Indeed, we have

   $$j_i = \ker p_i = \ker(j_{i-1} \circ p_i) = \ker d_i$$

   and similarly for the cokernel.

   In particular, the component short exact sequences of an acyclic chain complex are unique up to isomorphism.

2. If N is supported on $[n, m]$ then $Z_i = 0$ for every $i \geq n$, $Z_m = N_m$ and $Z_i = 0$ for every $i < m$.

3. An exact functor between exact categories preserves acyclic chain complexes, for it preserves a factorization into short exact sequences.

1.4 Exact categories that support long exact sequences

There is a technical condition that will be quite convenient to adopt:

Definition 1.4.1. Let $\mathcal{N}$ be an exact category. If there exists an admissible embedding $\mathcal{N} \hookrightarrow \mathcal{A}$ such that every long exact sequence in $\mathcal{A}$ whose objects are in $\mathcal{N}$ also has the kernels of its differentials in $\mathcal{N}$, we say that $\mathcal{N}$ *supports long exact sequences*. In this case, we will say that this embedding evidences the fact that $\mathcal{N}$ supports long exact sequences.

We will denote by $\mathbf{Exact}_{\mathbf{les}}$ the full subcategory of $\mathbf{Exact}$ whose objects are exact categories that support long exact sequences.

In the previous definition, we can replace “kernels” by “cokernels”, “images” or “coimages”.

Remark 1.4.2. There is a stronger condition appearing naturally. Let $\mathcal{N} \hookrightarrow \mathcal{A}$ be an admissible embedding. We say that $\mathcal{N}$ is *closed under kernels of epimorphisms* if whenever a map $f : N \to N''$ in $\mathcal{N}$ is an epimorphism in $\mathcal{A}$, we have that $\ker f$ is in $\mathcal{N}$.

We claim that a category closed under kernels of epimorphisms supports long exact sequences. Let (N, d) be a long exact sequence in $\mathcal{A}$ supported in $[n, m]$ whose objects are in $\mathcal{N}$. We have the following short exact sequence in $\mathcal{A}$:

$$\begin{tikzcd} 0 \arrow[r] & \ker d_{m+1} \arrow[r] & N_{m+1} \arrow[r, "d"] & N_m \arrow[r] & 0 \end{tikzcd}$$

Since $\mathcal{N}$ is closed under kernels of epimorphisms, we have that $\ker d_{m+1} \in \mathcal{N}$. We then proceed inductively: we consider the short exact sequence in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & \ker d_m \arrow[r] & N_m \arrow[r, "d"] & \ker d_{m+1} \arrow[r] & 0 \end{tikzcd}$$

whence $\ker d_m \in \mathcal{N}$, and so on.

Remark 1.4.3. Let $\mathcal{N}$ be an exact category that supports long exact sequences, as evidenced by an admissible embedding $\mathcal{N} \hookrightarrow \mathcal{A}$. Then the induced admissible embedding $C\mathcal{N} \hookrightarrow C\mathcal{A}$ shows that $C\mathcal{N}$ supports long exact sequences, too. This is true since kernels in $C\mathcal{N}$ are computed degreewise.

Example 1.4.4. The category $\mathrm{Proj}(R)$ is closed under kernels of epimorphisms, hence supports long exact sequences. Indeed, if $f : P \to P''$ is an epimorphism of finitely generated projective R-modules, then its kernel is a direct summand of P, hence projective. Observe that any finitely generated projective module is finitely presented. Therefore $\ker f$ is finitely generated, being the kernel of an epimorphism from a finitely generated projective module to a finitely presented one [10, (3.11), (3.13)].

1.5 Quasi-isomorphisms in $C\mathcal{N}$

Let $\mathcal{N}$ be an exact category.

We will now endow $C\mathcal{N}$ with a category of weak equivalences coarser than isomorphism that will turn it into a Waldhausen category if $\mathcal{N}$ supports long exact sequences.

Definition 1.5.1. Let $f : (N, d_N) \to (M, d_M)$ be a map in $C\mathcal{N}$.

- The *mapping cone* of f, denoted by $\mathrm{cone}(f)$, is the chain complex that has $N[-1] \oplus M$ as underlying graded object and $\begin{pmatrix} -d_N & 0 \\ f & d_M \end{pmatrix}$ as differential.
- The *mapping cylinder* of f, denoted by $\mathbb{T}(f)$, is the chain complex that has $N \oplus N[-1] \oplus M$ as graded object and $\begin{pmatrix} d_N & -\mathrm{id} & 0 \\ 0 & -d_N & 0 \\ 0 & f & d_M \end{pmatrix}$ as differential. The inclusion on the first factor $\lambda : N \to$

$\mathbb{T}(f)$ will be called the *front inclusion*, the inclusion on the third factor $\rho : M \to \mathbb{T}(f)$ will be called the *back inclusion*, and the map $\pi = \begin{pmatrix} f & 0 & \mathrm{id}_M \end{pmatrix} : \mathbb{T}(f) \to M$ will be called the *projection*.

Remark 1.5.2.

1. If $f : N \to M$ is chain map, there is a short exact sequence in $C\mathcal{N}$:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & \mathrm{cone}(f) \arrow[r] & N[-1] \arrow[r] & 0 \end{tikzcd}$$

where the first map is inclusion on the second summand and the second map is minus the projection on the first summand.

2. If $F : \mathcal{N} \to \mathcal{M}$ is an arrow in **Exact**, then a straightforward computation shows that $CF : C\mathcal{N} \to C\mathcal{M}$ preserves mapping cones: $(CF)(\mathrm{cone}(f)) = \mathrm{cone}((CF)(f))$.

Definition 1.5.3. We say that a chain map $f : N \to M$ is a *quasi-isomorphism* if $\mathrm{cone}(f)$ is acyclic.

In an abelian category, we usually define a quasi-isomorphism to be a chain map that is an isomorphism in homology, but this is equivalent to having an acyclic cone ([15, (1.5.4)]), thus the two definitions are compatible.

Remark 1.5.4. From remarks 1.5.2.2 and 1.3.3.3 we see that if $F : \mathcal{N} \to \mathcal{M}$ is an arrow in **Exact**, then $CF : C\mathcal{N} \to C\mathcal{M}$ preserves quasi-isomorphisms.

We now record some useful consequences that the condition of supporting long exact sequences has in the quasi-isomorphisms.

Remark 1.5.5. Suppose $\mathcal{N}$ supports long exact sequences, as evidenced by an admissible embedding $\mathcal{N} \hookrightarrow \mathcal{A}$.

1. A chain complex in $C\mathcal{N}$ is acyclic in $C\mathcal{N}$ if and only if it is acyclic in $C\mathcal{A}$. The very definition of supporting long exact sequences is engineered in a way that this assertion is true. This observation implies the following ones.

2. A chain map in $C\mathcal{N}$ is a quasi-isomorphism in $C\mathcal{N}$ if and only if it is a quasi-isomorphism in $C\mathcal{A}$.

3. In a short exact sequence in $C\mathcal{N}$, if two of the terms are acyclic then so is the third (for abelian categories, this is the diagram chase in [15, Ex. (1.3.1)]).

4. Any map between acyclic chain complexes in $C\mathcal{N}$ is a quasi-isomorphism. In particular, if N is an acyclic chain complex, then the maps $0 \to N$ and $N \to 0$ are quasi-isomorphisms.

Lemma 1.5.6. *Let $\mathcal{A}$ be an abelian category and consider a morphism between short exact sequences in $C\mathcal{A}$:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

If two of α, β, γ are quasi-isomorphisms, then so is the third.

Proof. Suppose α, β are quasi-isomorphisms. Then γ is a quasi-isomorphism by the five lemma, as shown in the following commutative ladder of long exact sequences of homology:

$$\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d, "\cong"'] & H_n(B) \arrow[r] \arrow[d, "\cong"'] & H_n(C) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d, "\cong"'] & H_{n-1}(B) \arrow[d, "\cong"'] \\
H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r] & H_{n-1}(A') \arrow[r] & H_{n-1}(B')
\end{tikzcd}$$

The other two cases follow analogously. □

Proposition 1.5.7. *Suppose $\mathcal{N}$ supports long exact sequences. The quasi-isomorphisms in $C\mathcal{N}$ are a lluf subcategory $q \subset C\mathcal{N}$ such that $qC\mathcal{N}$ is a Waldhausen category that satisfies the saturation and extension axioms, and the mapping cylinder defines a cylinder functor that satisfies the cylinder axiom and such that the back inclusion is a quasi-isomorphism.*

Proof. We keep as cofibrations the admissible monomorphisms (remark 1.2.4). We first check that $qC\mathcal{N}$ is actually a category of weak equivalences.

Let $\mathcal{N} \hookrightarrow \mathcal{A}$ be an admissible embedding of $\mathcal{N}$ into an abelian category that shows that $\mathcal{N}$ supports long exact sequences, so we can freely use remark 1.5.5.2: a chain map in $C\mathcal{N}$ is a quasi-isomorphism in $C\mathcal{N}$ if and only if it is so in $C\mathcal{A}$.

- q is a subcategory of $C\mathcal{N}$. We need to check that identities are quasi-isomorphisms and that the composition of quasi-isomorphisms is a quasi-isomorphism. This is trivial in $C\mathcal{A}$, by functoriality of homology.
- The isomorphisms of $C\mathcal{N}$ are quasi-isomorphisms. Again, functoriality of homology gives the result.
- We will now prove that the gluing axiom is satisfied. Consider the following cube in $C\mathcal{N}$:

$$\begin{tikzcd}
& B \arrow[rr, "h"] \arrow[dd, "\beta"', "\simeq"] & & P \arrow[dd, dotted, "r"] \\
A \arrow[ur, tail, "f"] \arrow[rr, "g"] \arrow[dd, "\alpha"', "\simeq"] & & C \arrow[ur, tail, "i"'] \arrow[dd, "\gamma"', "\simeq"] & \\
& B' \arrow[rr, "h'"] & & P' \\
A' \arrow[ur, tail, "f'"'] \arrow[rr, "g'"'] & & C' \arrow[ur, tail, "i'"'] &
\end{tikzcd}$$

  where $P = B \cup_A C$, $P' = B' \cup_{A'} C'$ and r is the induced arrow, which we need to prove is a quasi-isomorphism.

  We now work in $C\mathcal{A}$, where we have homology at our disposal.

  Let $E = \operatorname{coker} f$, $E' = \operatorname{coker} f'$ and e be the induced map as shown in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"', "\simeq"] & B \arrow[r] \arrow[d, "\beta"', "\simeq"] & E \arrow[r] \arrow[d, dotted, "e"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r] & E' \arrow[r] & 0
\end{tikzcd}$$

Similarly, let $L = \operatorname{coker} i$, $L' = \operatorname{coker} i'$ and $\ell : L \to L'$ be the induced map as shown in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "\gamma"', "\simeq"] & P \arrow[r] \arrow[d, "r"'] & L \arrow[r] \arrow[d, dotted, "\ell"'] & 0 \\
0 \arrow[r] & C' \arrow[r, "i'"'] & P' \arrow[r] & L' \arrow[r] & 0
\end{tikzcd}$$

By lemma 1.5.6 it follows that e is a quasi-isomorphism. By the same lemma, to prove that r is a quasi-isomorphism it suffices to see that ℓ is one.

Let s be the following induced arrow on cokernels:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r] \arrow[d, "h"'] & E \arrow[r] \arrow[d, dotted, "s"'] & 0 \\
0 \arrow[r] & C \arrow[r, "i"'] & P \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

Recall that in an abelian category, the cokernels of a pair of parallel arrows in a pushout are compatibly isomorphic. Thus s is an isomorphism. Putting primes over every letter in the last diagram, we obtain an arrow $s' : E' \to L'$ which is also an isomorphism.

A straightforward diagram chase proves that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "s"] \arrow[d, "e"'] & L \arrow[d, "\ell"] \\
E' \arrow[r, "s'"'] & L'
\end{tikzcd}$$

Taking homology we have the following commutative diagram:

$$\begin{tikzcd}
H_n(E) \arrow[r, "s_*", "\cong"'] \arrow[d, "e_*"', "\cong"] & H_n(L) \arrow[d, "\ell_*"] \\
H_n(E') \arrow[r, "\cong", "s'_*"'] & H_n(L')
\end{tikzcd}$$

Thus ℓ_* is an isomorphism for all n, therefore ℓ is a quasi-isomorphism.

- The saturation axiom follows from functoriality of homology.
- The extension axiom follows from lemma 1.5.6.

- We now check that the mapping cylinder verifies the cylinder conditions and its axiom. First observe that the cylinder construction is functorial: if we have the following morphism of chain maps,

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "\alpha"'] & M \arrow[d, "\beta"] \\ N' \arrow[r, "f'"'] & M' \end{tikzcd}$$

we define $\mathbb{T}(f) \to \mathbb{T}(f')$ as $\left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{smallmatrix}\right)$, where the α in the middle is to be understood as $\alpha[-1]$, a slight abuse of notation we will allow ourselves to incur in more than once. It is immediate that identities and compositions are respected.

We now verify the conditions from definition 1.2.5:

(i) The following diagram commutes by construction:

$$\begin{tikzcd} N \arrow[r, "\lambda"] \arrow[dr, "f"'] & \mathbb{T}(f) \arrow[d, "\pi"] & M \arrow[l, "\rho"'] \arrow[dl, "\mathrm{id}"] \\ & M & \end{tikzcd}$$

(ii) $\mathbb{T}(0 \to N) \cong N$ and its back inclusion and projection are identities under this identification.

(iii) $j_1 \oplus j_2$ is an admissible monomorphism since there is a short exact sequence in $C\mathcal{N}$:

$$\begin{tikzcd} 0 \arrow[r] & N \oplus M \arrow[r, "j_1 \oplus j_2"] & \mathbb{T}(f) \arrow[r, "(0\ \mathrm{id}\ 0)"] & N[-1] \arrow[r] & 0 \end{tikzcd}$$

(iv) The induced map $\mathbb{T}(f) \to \mathbb{T}(f')$ is a quasi-isomorphism if α and β are. To see this, consider the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & N \oplus M \arrow[r, "j_1 \oplus j_2"] \arrow[d, "\alpha\oplus\beta"'] & \mathbb{T}(f) \arrow[r, "(0\ \mathrm{id}\ 0)"] \arrow[d] & N[-1] \arrow[r] \arrow[d, "\alpha"'] & 0 \\ 0 \arrow[r] & N' \oplus M' \arrow[r, "j_1' \oplus j_2'"'] & \mathbb{T}(f') \arrow[r, "(0\ \mathrm{id}\ 0)"'] & N'[-1] \arrow[r] & 0 \end{tikzcd}$$

The right vertical map is a quasi-isomorphism, and since homology preserves finite direct sums, the left vertical map is one too. Applying lemma 1.5.6 we get that the map $\mathbb{T}(f) \to \mathbb{T}(f')$ is a quasi-isomorphism.

(v) Suppose now that α and β are admissible monomorphisms. First, let us check that $\mathbb{T}(f) \to \mathbb{T}(f')$ is one too. The maps α and β fit into short exact sequences as in the following commutative diagram, where f'' is the induced map in cokernels:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "f"'] & N' \arrow[r, "\alpha'"] \arrow[d, "f'"'] & N'' \arrow[r] \arrow[d, "f''"'] & 0 \\ 0 \arrow[r] & M \arrow[r, "\beta"'] & M' \arrow[r, "\beta'"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

Let C be the chain complex having $N'' \oplus N''[-1] \oplus M''$ as underlying graded object, and $\begin{pmatrix} d & -\mathrm{id} & 0 \\ 0 & -d & 0 \\ 0 & f'' & d \end{pmatrix}$ as differential. It is readily checked that the square of this matrix map is zero. Consider the following sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{T}(f) \arrow[r, "\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}"] & \mathbb{T}(f') \arrow[r, "\begin{pmatrix} \alpha' & 0 & 0 \\ 0 & \alpha' & 0 \\ 0 & 0 & \beta' \end{pmatrix}"] & C \arrow[r] & 0
\end{tikzcd}$$

The second map is indeed a chain map because the differential of C was thus engineered. Now, the forgetful functor $C\mathcal{N} \to \mathrm{Gr}\mathcal{N}$ that "forgets the differential" reflects exactness, by definition of the exact structures of $C\mathcal{N}$ and $\mathrm{Gr}\mathcal{N}$. This proves that the above sequence is short exact in $C\mathcal{N}$, since it is so in $\mathrm{Gr}\mathcal{N}$, being a direct sum of short exact sequences (proposition [1, (2.9)]).

Let us now identify the following pushout:

$$\begin{tikzcd}
N \oplus M \arrow[r, tail, "j_1 \oplus j_2"] \arrow[d, tail, "\alpha\oplus\beta"'] & \mathbb{T}(f) \arrow[d] \\
N' \oplus M' \arrow[r] & P
\end{tikzcd} \tag{1.3}$$

Let A be the chain complex having $N' \oplus N[-1] \oplus M'$ as underlying graded object, and $\begin{pmatrix} 0 & -d & 0 \\ d & -\alpha & 0 \\ 0 & \beta f & d \end{pmatrix}$ as differential. It is readily checked that the square of this matrix map is zero. Consider the following diagram.

$$\begin{tikzcd}
N \oplus M \arrow[r, tail, "j_1 \oplus j_2"] \arrow[d, tail, "\alpha\oplus\beta"'] & \mathbb{T}(f) \arrow[r] \arrow[d, "\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \beta \end{pmatrix}"'] & N[-1] \arrow[d, "\mathrm{id}"'] \\
N' \oplus M' \arrow[r, "\begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}"'] & A \arrow[r] & N[-1]
\end{tikzcd} \tag{1.4}$$

It is indeed a commutative diagram in $C\mathcal{N}$: the differential in A is engineered so as to make the displayed matrix maps $N' \oplus M' \to A$ and $\mathbb{T}(f) \to A$ commute with the differentials.

Its top and bottom row are short exact sequences: this follows by lemma [1, (2.7)]), using the fact that the forgetful functor $C\mathcal{N} \to \mathrm{Gr}\mathcal{N}$ reflects exactness. Therefore the left square of (1.4) is a pushout (proposition [1, (2.12)]).

Having identified the pushout in (1.3), we will now identify the induced arrow h making the following diagram commute, and then prove that it is an admissible monomorphism.

$$\begin{tikzcd}
N \oplus M \arrow[r, tail, "j_1 \oplus j_2"] \arrow[d, tail, "\alpha \oplus \beta"'] & \mathbb{T}(f) \arrow[d, "\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \beta \end{pmatrix}"] \arrow[rdd, bend left, "\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}"] & \\
N' \oplus M' \arrow[r, "\begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}"'] \arrow[rrd, bend right, "j_1' \oplus j_2'"'] & A \arrow[rd, dotted, "h"] & \\
& & \mathbb{T}(f')
\end{tikzcd} \tag{1.5}$$

It is immediate that $h = \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}$, since this matrix map indeed commutes with the differentials and makes diagram (1.5) commute.

Consider the following sequence:

$$0 \longrightarrow A \xrightarrow{h} \mathbb{T}(f') \xrightarrow{(0\ \alpha'\ 0)} N''[-1] \longrightarrow 0$$

Once again, since the forgetful functor $C\mathcal{N} \to \mathrm{Gr}\mathcal{N}$ reflects exactness, we see that it is a short exact sequence in $C\mathcal{N}$.

This proves that h is an admissible monomorphism.

Let us check that the back inclusion is a quasi-isomorphism. We have an obvious short exact sequence in $C\mathcal{N}$:

$$0 \longrightarrow M \xrightarrow{\rho} \mathbb{T}(f) \longrightarrow \mathrm{cone}(-\mathrm{id}_N) \longrightarrow 0$$

Considering its long exact sequence in homology, since $-\mathrm{id}_N$ is a (quasi)-isomorphism we see that ρ is a quasi-isomorphism.

It now follows that π is a quasi-isomorphism and thus the cylinder axiom is satisfied. Indeed, functoriality of homology applied to the equality $\pi\rho = \mathrm{id}_M$ yields the result. ☐

Remark 1.5.8. Remark 1.5.4 gives that if $F : \mathcal{N} \to \mathcal{M}$ is an arrow in $\mathbf{Exact}_{\mathbf{les}}$, then $CF : C\mathcal{N} \to C\mathcal{M}$ verifies $(CF)(q) \subset q$. We thus have an obvious functor $qC : \mathbf{Exact}_{\mathbf{les}} \to \mathbf{Wald}$.

Corollary 1.5.9. *Suppose $\mathcal{N}$ supports long exact sequences. Then $C^q\mathcal{N} := (C\mathcal{N})^q$ is an exact category that supports long exact sequences.*

Proof. The category $C^q\mathcal{N}$ is exact by virtue of proposition 1.2.7. It has kernels of epimorphisms, thanks to remark 1.5.5.3, thus it supports long exact sequences (remark 1.4.2). ☐

Remark 1.5.10.
1. Since exact functors between exact categories preserve quasi-isomorphisms (remark 1.5.4), we have a functor $C^q : \mathbf{Exact}_{\mathbf{les}} \to \mathbf{Exact}_{\mathbf{les}}$.
2. By way of remark 1.5.5.2 we see that the category $C^q\mathcal{N}$ is the full subcategory of $C\mathcal{N}$ consisting of acyclic chain complexes.

1.6 Binary chain complexes

Let $\mathcal{N}$ be an exact category.

Definition 1.6.1. A *binary chain complex* in $\mathcal{N}$ is a chain complex in $\mathcal{N}$ with two differentials. More formally, it is a triple (N, d, d') where $N \in \mathrm{Gr}\mathcal{N}$ and d, d' are differentials on N. We say that d is the *top differential* and that d' is the *bottom differential*. We say (N, d, d') is *bounded* if N is.

We will sometimes write N to mean a binary chain complex, not specifiying its differentials. In this case we will write $\mathrm{gr}N$ to mean the underlying graded object.

If $n > m$ are integers, we say that a binary chain complex N is *supported on the interval* $[n, m]$ if $N_i = 0$ for all $i > n$ and $i < m$, and we say it has *length* $n - m$.

A *morphism* between two binary chain complexes is a map between the underlying graded objects that commutes with both differentials.

We will denote by $B\mathcal{N}$ the category of bounded binary chain complexes.

Convention 1.6.2. We will make the convention that binary chain complexes be bounded.

Definition 1.6.3. The *diagonal functor* is the functor $\Delta : C\mathcal{N} \to B\mathcal{N}$ that maps (N, d) to (N, d, d) objectwise and is defined obviously on morphisms.

The *top* and *bottom* functors $\top, \bot : B\mathcal{N} \to C\mathcal{N}$ map (N, d, d') to (N, d) and (N, d') respectively, and are defined obviously on morphisms.

We say that a short sequence in $B\mathcal{N}$ is a *short exact sequence*, or that a binary chain complex is *acyclic*, if and only if its images under the functors $\top$ and $\bot$ have the same property.

This turns $B\mathcal{N}$ into an exact category and $\Delta, \top, \bot$ into exact functors.

If $\mathcal{N}$ supports long exact sequences, then so does $B\mathcal{N}$, just like in remark 1.4.3.

Definition 1.6.4. We define three lluf subcategories of $B\mathcal{N}$:

- The category q, whose morphisms (called *quasi-isomorphisms*) are the ones such that their images under $\top$ and $\bot$ are quasi-isomorphisms in $C\mathcal{N}$,
- The category t, whose morphisms are the ones such that their image under $\top$ is a quasi-isomorphism in $C\mathcal{N}$,
- The category b, whose morphisms are the ones such that their image under $\bot$ is a quasi-isomorphism in $C\mathcal{N}$.

Let us observe how all of these actors fit together:

Remark 1.6.5. 1. The categories $qB\mathcal{N}$, $tB\mathcal{N}$ and $bB\mathcal{N}$ are Waldhausen categories that satisfy the saturation and extension axioms, and the functors $\Delta : qC\mathcal{N} \to qB\mathcal{N}$, $\top, \bot : qB\mathcal{N} \to qC\mathcal{N}$, $\top : tB\mathcal{N} \to qC\mathcal{N}$, $\bot : bB\mathcal{N} \to qC\mathcal{N}$ are exact. The verifications of these assertions are completely analogous to the ones for $qC\mathcal{N}$.

We can also create through $\top$ and $\bot$ a cylinder functor in these three categories that satisfies the cylinder axiom. To illustrate, if $f : (N, d, d') \to (M, e, e')$ is an arrow in $qB\mathcal{N}$, define $\mathbb{T}(f) \in qB\mathcal{N}$ as the binary chain complex having $N \oplus N[-1] \oplus M$ as underlying graded object, $\begin{pmatrix} d & -\mathrm{id} & 0 \\ 0 & -d & 0 \\ 0 & f & e \end{pmatrix}$ as top differential and $\begin{pmatrix} d' & -\mathrm{id} & 0 \\ 0 & -d' & 0 \\ 0 & f & e' \end{pmatrix}$ as bottom differential. This defines a cylinder functor in $qB\mathcal{N}$ together with the projection and front and back inclusions we have already considered for $qC\mathcal{N}$, since they commute with both differentials. Observe that $\top\mathbb{T}(f) = \mathbb{T}(\top f)$ and $\bot\mathbb{T}(f) = \mathbb{T}(\bot f)$.

2. Define $B^q\mathcal{N} := (B\mathcal{N})^q$, $B^t\mathcal{N} := (B\mathcal{N})^t$ and $B^b\mathcal{N} := (B\mathcal{N})^b$ as per proposition 1.2.7. If $\mathcal{N}$ supports long exact sequences, then $B^q\mathcal{N}$, $B^t\mathcal{N}$ and $B^b\mathcal{N}$ are exact categories that support long exact sequences, whose objects are respectively the binary chain complexes with both differentials acyclic, which we call *acyclic binary chain complexes*, the binary chain complexes with acyclic top differential, and the binary chain complexes with acyclic bottom differential, just as we saw in corollary 1.5.9 and remark 1.5.10.2 for $C^q\mathcal{N}$.

3. The categories $B^t\mathcal{N}$ and $B^b\mathcal{N}$ also admit the quasi-isomorphisms q as a category of weak equivalences. We have then Waldhausen categories $qB^t\mathcal{N}$ and $qB^b\mathcal{N}$. We also have categories of weak equivalences $b \subset B^t\mathcal{N}$ and $t \subset B^b\mathcal{N}$. But observe that $tB^b\mathcal{N} = qB^b\mathcal{N}$ and $bB^t\mathcal{N} = qB^t\mathcal{N}$. Indeed, the reverse inclusion is obvious, and the forward one follows because any morphism between acyclic chain complexes is a quasi-isomorphism (remark 1.5.5.4).

4. Observe that $\Delta : C^q\mathcal{N} \to B^q\mathcal{N}$, $\top, \bot : B^q\mathcal{N} \to C^q\mathcal{N}$, $\top : B^t\mathcal{N} \to C^q\mathcal{N}$ and $\bot : B^b\mathcal{N} \to C^q\mathcal{N}$ are exact functors, since indeed they map acyclic (binary) chain complexes to acyclic (binary) chain complexes.

Some functoriality and naturality remarks are in order:

Remark 1.6.6. 1. Functoriality remarks for binary chain complexes are similar to the ones we did for chain complexes. We have a functor $B : \mathbf{Exact} \to \mathbf{Exact}$. If $F : \mathcal{N} \to \mathcal{M}$ is an exact functor between exact categories, then BF preserves quasi-isomorphisms. We also have functors $qB, tB, bB : \mathbf{Exact_{les}} \to \mathbf{Wald}$, $B^q, B^t, B^b : \mathbf{Exact_{les}} \to \mathbf{Exact_{les}}$ and $qB^b = tB^b, qB^t = bB^t : \mathbf{Exact_{les}} \to \mathbf{Wald}$.

2. We have three different ways to exhibit Δ as a natural transformation:

$$\begin{tikzcd} \mathbf{Exact} \arrow[r, bend left, "C"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow\Delta"] & \mathbf{Exact} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Exact_{les}} \arrow[r, bend left, "qC"] \arrow[r, bend right, "qB"'] \arrow[r, phantom, "\Downarrow q\Delta"] & \mathbf{Wald} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Exact_{les}} \arrow[r, bend left, "C^q"] \arrow[r, bend right, "B^q"'] \arrow[r, phantom, "\Downarrow\Delta^q"] & \mathbf{Exact_{les}} \end{tikzcd}$$

If $F : \mathcal{N} \to \mathcal{M}$ is an arrow in $\mathbf{Exact}$ for the first case and in $\mathbf{Exact_{les}}$ for the second and third cases, we have three commutative squares:

$$\begin{tikzcd} C\mathcal{M} \arrow[r, "\Delta"] \arrow[d, "CF"'] & B\mathcal{M} \arrow[d, "BF"] \\ C\mathcal{N} \arrow[r, "\Delta"'] & B\mathcal{N} \end{tikzcd} \qquad \begin{tikzcd} qC\mathcal{M} \arrow[r, "q\Delta"] \arrow[d, "qCF"'] & qB\mathcal{M} \arrow[d, "qBF"] \\ qC\mathcal{N} \arrow[r, "q\Delta"'] & qB\mathcal{N} \end{tikzcd} \qquad \begin{tikzcd} C^q\mathcal{M} \arrow[r, "\Delta^q"] \arrow[d, "C^qF"'] & B^q\mathcal{M} \arrow[d, "B^qF"] \\ C^q\mathcal{N} \arrow[r, "\Delta^q"'] & B^q\mathcal{N} \end{tikzcd}$$

We will sometimes denote any of these three natural transformations merely by Δ. There is no danger of confusion if domain and codomain are specified.

3. As for $\top$ and $\bot$, we can exhibit them like this:

$$\begin{tikzcd} \mathbf{Exact} \arrow[r, bend left, "B"] \arrow[r, bend right, "C"'] \arrow[r, phantom, "\Downarrow \top{,}\bot"] & \mathbf{Exact} \end{tikzcd} \quad \begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "qB"] \arrow[r, bend right, "qC"'] \arrow[r, phantom, "\Downarrow q\top{,}q\bot"] & \mathbf{Wald} \end{tikzcd} \quad \begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "B^q"] \arrow[r, bend right, "C^q"'] \arrow[r, phantom, "\Downarrow \top^q{,}\bot^q"] & \mathbf{Exact}_{\mathbf{les}} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "tB"] \arrow[r, bend right, "qC"'] \arrow[r, phantom, "\Downarrow t\top"] & \mathbf{Wald} \end{tikzcd} \quad \begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "B^t"] \arrow[r, bend right, "C^q"'] \arrow[r, phantom, "\Downarrow \top^t"] & \mathbf{Exact}_{\mathbf{les}} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "bB"] \arrow[r, bend right, "qC"'] \arrow[r, phantom, "\Downarrow b\bot"] & \mathbf{Wald} \end{tikzcd} \quad \begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "B^b"] \arrow[r, bend right, "C^q"'] \arrow[r, phantom, "\Downarrow \bot^b"] & \mathbf{Exact}_{\mathbf{les}} \end{tikzcd}$$

   We will sometimes denote any of these merely by $\top$ or $\bot$. There is no danger of confusion if domain and codomain are specified.

4. Recall that specifying a natural transformation $\begin{tikzcd} \mathcal{C} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \end{tikzcd}$ is the same as specifying a functor $\mathcal{C} \to \operatorname{Arr} \mathcal{D}$. We will use this observation more than once for the natural transformations above.

Definition 1.6.7. Suppose we are given $N, P \in B\mathcal{N}$ and chain maps $r : \top N \to \top P$ and $r' : \bot N \to \bot P$. We define the *binary mapping cylinder* $\mathbb{T}(r, r') \in B\mathcal{N}$ of r and r' as follows.

Observe that the mapping cylinders $\mathbb{T}(r)$ and $\mathbb{T}(r')$ have the same underlying graded object, namely $\mathrm{gr}N \oplus \mathrm{gr}N[-1] \oplus \mathrm{gr}P$. Thus we can paste $\mathbb{T}(r)$ and $\mathbb{T}(r')$ into a binary chain complex, with top (resp. bottom) differential given by the differential of $\mathbb{T}(r)$ (resp. $\mathbb{T}(r')$).

The two front inclusions $\top N \to \mathbb{T}(r)$ and $\bot N \to \mathbb{T}(r')$ agree on underlying graded objects (they are just inclusions in the first variable), thus they give a map $\lambda : N \to \mathbb{T}(r, r')$ which we also call *front inclusion*.

We similarly define the *back inclusion*.

Remark 1.6.8. 1. This is not the mapping cylinder of $qB\mathcal{N}$ considered in 1.6.5.1, and it cannot be a cylinder in the sense of definition 1.2.5. Indeed, such a cylinder is defined for a map between binary chain complexes, whereas this "binary mapping cylinder" is defined for two maps between the underlying chain complexes of a binary chain complex.

2. There is no sensible way to define a "projection" for the binary mapping cylinder in the vein of definition 1.2.5, since the projections of $\mathbb{T}(r)$ and $\mathbb{T}(r')$ do not agree on underlying graded objects, as they depend on r and r'.

3. $\top\mathbb{T}(r, r') = \mathbb{T}(r)$ and $\bot\mathbb{T}(r, r') = \mathbb{T}(r')$.

4. The back inclusion is a quasi-isomorphism, because it is so for the back inclusions in $qC\mathcal{N}$ (proposition 1.5.7).

1.7 K-theory of Waldhausen categories

From this section on we will be dealing with spectra. For us, the category **Spectra** of spectra is taken to mean the category of CW-spectra.

There is a functor K from the category of Waldhausen categories to the category of connective Ω-spectra. If $i \geq 0$, the composition with the π_i functor from the category of spectra to the category of abelian groups gives the functor $K_i : \mathbf{Wald} \to \mathbf{Ab}$.

The construction of the functor K can be found in [14, (1.3)] or in [16, (IV.8)], for example; we will not be using it.

One advantage of working with K-theory spectra instead of spaces is the fact that (homotopy) fibrations and cofibrations coincide. We will use this convenient fact more than once.

We will also be using relative K-theory. Let $F : w\mathcal{C} \to v\mathcal{D}$ be an exact functor between Waldhausen categories. There is a *relative K-theory* connective spectrum $K[F]$ that fits into a homotopy fibration sequence as follows:

$$\begin{tikzcd} Kw\mathcal{C} \arrow[r, "KF"] & Kv\mathcal{D} \arrow[r] & K[F] \end{tikzcd}$$

There is an explicit construction of $K[F]$, of the map $Kv\mathcal{D} \to K[F]$ and of a nullhomotopy of the composition in [14, (1.5.7)]; these are summarized in [4, (A.4)].

This construction is functorial: there is a functor $K : \mathrm{Arr}\,(\mathbf{Wald}) \to \mathbf{Spectra}$ that maps a commutative square of exact categories to the induced map between homotopy cofibers of their induced maps in K-theory, presented in the following commutative diagram:

$$\begin{tikzcd} Kw\mathcal{C} \arrow[r, "KF"] \arrow[d] & Kv\mathcal{D} \arrow[r] \arrow[d] & K[F] \arrow[d, dotted] \\ Kw'\mathcal{C}' \arrow[r, "KG"] & Kv'\mathcal{D}' \arrow[r] & K[G] \end{tikzcd}$$

Remark 1.7.1. There are K-functors $K : \mathbf{Exact} \to \mathbf{Spectra}$ and $K : \mathrm{Arr}\,(\mathbf{Exact}) \to \mathbf{Spectra}$, defined by the inclusion $\mathbf{Exact} \hookrightarrow \mathbf{Wald}$ that considers an exact category as a Waldhausen category with the isomorphisms as quasi-isomorphisms (example 1.2.4). We can also take them to be Quillen's K-theory defined for exact categories, see [14, (1.9)] for a proof that both of these coincide in this framework.

Definition 1.7.2. Given a category $\mathcal{C}$ and two functors $F, G : \mathcal{C} \to \mathbf{Spectra}$, we define a *natural homotopy equivalence* $\tau : F \Rightarrow G$ to be a family $\{\tau_C : FC \to GC\}_{C\in\mathcal{C}}$ of homotopy equivalences in **Spectra** such that the following square is homotopy commutative for any $f : C \to C'$ in $\mathcal{C}$:

$$\begin{tikzcd} FC \arrow[r, "\tau_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "\tau_{C'}"'] & GC' \end{tikzcd}$$

A *natural zigzag homotopy equivalence* between F and G is defined as a finite sequence of natural homotopy equivalences having F and G as their ends and going in any direction.

For example, $F \Rightarrow H \Leftarrow G$ is a natural zigzag homotopy equivalence between F and G provided both double arrows are natural homotopy equivalences.

We will denote a natural zigzag homotopy equivalence from F to G as $F \overset{\simeq}{\rightsquigarrow} G$ or as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\wr\simeq"] & \mathbf{Spectra} \end{tikzcd} .$$

Remark 1.7.3. Given a natural zigzag homotopy equivalence as above and a functor $P : \mathbf{Spectra} \to \mathbf{Ab}$ that descends to the homotopy category of spectra (such as the homotopy functors), composing them we get a natural isomorphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "PF"] \arrow[r, bend right, "PG"'] \arrow[r, phantom, "\Downarrow\cong"] & \mathbf{Ab} \end{tikzcd} .$$

Indeed, homotopy equivalences are mapped to isomorphisms which are invertible maps, so that we get honest arrows $PFC \to PGC$ for any $C \in \mathcal{C}$, and for the same reason homotopy commutative squares yield naturality squares.

We now cite the properties we will be using.

Proposition 1.7.4. *Let $w\mathcal{C}$ be a Waldhausen category. Then $K_0(w\mathcal{C})$ is the abelian group with the following presentation: it has one generator $[C]$ per object $C \in \mathcal{C}$, and these are subject to the relations*

- $[C] = [C']$ *if there is a weak equivalence $C \to C'$,*
- $[C] = [B] + [C/B]$ *for every cofibration sequence* $B \rightarrowtail C \twoheadrightarrow B/C$

Moreover, if $F : w\mathcal{C} \to v\mathcal{D}$ is an exact functor between Waldhausen categories, then $K_0F : K_0(w\mathcal{C}) \to K_0(v\mathcal{D})$ maps $[C]$ to $[FC]$ for every $C \in \mathcal{C}$.

Observe that if $\mathcal{C}$ is actually an exact category and w are the isomorphisms, then the first condition follows from the second one.

For a proof, see [16, IV.8.4].

Proposition 1.7.5 (Compatibility with finite products and filtered colimits)**.** *If $\mathcal{N}, \mathcal{M}$ are exact categories, then $K(\mathcal{N} \times \mathcal{M}) \cong K\mathcal{N} \times K\mathcal{M}$.*

*Let J be a small filtered category. Let $J \to$ **Exact**, $i \mapsto \mathcal{N}_i$ be a functor. Then the colimit $\varinjlim \mathcal{N}_i \in$ **Exact** exists, and $K(\varinjlim \mathcal{N}_i) = \varinjlim(K\mathcal{N}_i)$. In particular $K_n(\varinjlim \mathcal{N}_i) = \varinjlim K_n(\mathcal{N}_i)$ for every $n \geq 0$.*

For a proof, see [16, p. 351].

Definition 1.7.6. Let $F, F', F'' : \mathcal{N} \to \mathcal{M}$ be exact functors between exact categories. Suppose we have a sequence of natural transformations $F' \to F \to F''$. We say that it is a *short exact sequence* and we write $0 \to F' \to F \to F'' \to 0$ if for every $N \in \mathcal{N}$ the induced sequence $0 \to F'N \to FN \to F''N \to 0$ is short exact in $\mathcal{M}$.

Theorem 1.7.7 (Additivity theorem)**.** *If $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of exact functors between exact categories, then there is a homotopy of maps of spectra $KF \simeq KF' + KF''$.*

For a proof, see [16, (V.1.2)], [14, (1.3.2), (1.4.2)] or [12, (1.7.2)]. There's also an analogue for general Waldhausen categories but we won't be needing it.

Definition 1.7.8. Let $F, G : w\mathcal{C} \to v\mathcal{D}$ be exact functors between Waldhausen categories. We say that a natural transformation $\tau : F \Rightarrow G$ is a *weak equivalence* if for every $C \in \mathcal{C}$ the map $\tau_C : FC \to GC$ is a weak equivalence in $v\mathcal{D}$.

Proposition 1.7.9. *A weak equivalence $F \Rightarrow G$ induces a homotopy of maps from KF to KG.*

For a proof, see [14, (1.3.1)].

Theorem 1.7.10 (Waldhausen's localization theorem)**.** *Let $\mathcal{C}$ be a category with cofibrations, and let $v, w \subset \mathcal{C}$ be categories of weak equivalences in $\mathcal{C}$ with $v \subset w$. Suppose $w\mathcal{C}$ satisfies the saturation and extension axioms, and it has a cylinder functor that satisfies the cylinder axiom. Then the inclusions $v\mathcal{C}^w \to v\mathcal{C}$ and $v\mathcal{C} \to w\mathcal{C}$ induce a homotopy fibration sequence:*

$$Kv\mathcal{C}^w \longrightarrow Kv\mathcal{C} \longrightarrow Kw\mathcal{C}$$

Here the specified nullhomotopy for the composition is given as follows. If $i : v\mathcal{C}^w \to w\mathcal{C}$ is the inclusion functor, then we have a weak equivalence $0 \Rightarrow i$ and therefore by 1.7.9 a nullhomotopy of Ki.

For a proof, see [14, (1.6.4)], [12, (1.8.2)] or [16, (V.2.1)].

Theorem 1.7.11 (Thomason's cofinality theorem)**.** *Let $v\mathcal{C}$ be a Waldhausen category with a cylinder functor satisfying the cylinder axiom. Let G be an abelian group and $\pi : K_0 v\mathcal{C} \to G$ an epimorphism. Let $\mathcal{C}_\pi$ be the full subcategory of $\mathcal{C}$ whose objects are those $C \in \mathcal{C}$ for which the class $[C] \in K_0 v\mathcal{C}$ is in $\ker \pi$. Then $\mathcal{C}_\pi$ is a Waldhausen category with cofibrations (resp. weak equivalences) the cofibrations (resp. weak equivalences) those of $\mathcal{C}$ which are in $\mathcal{C}_\pi$. Denote also by v the subcategory of weak equivalences of $\mathcal{C}_\pi$.*

Let "G" denote the Eilenberg-Mac Lane spectrum of G whose only non-zero homotopy group is G in degree 0. Then there is a homotopy fibration sequence of spectra:

$$K(v\mathcal{C}_\pi) \to K(v\mathcal{C}) \to \text{“}G\text{”}$$

where the first map is induced by inclusion.

In particular, the map $K(v\mathcal{C}_\pi) \to K(v\mathcal{C})$ induces isomorphisms $K_i(v\mathcal{C}_\pi) \cong K_i(v\mathcal{C})$ for $i > 0$ and there is a short exact sequence of abelian groups:

$$0 \longrightarrow K_0(v\mathcal{C}_\pi) \longrightarrow K_0(v\mathcal{C}) \xrightarrow{\pi} G \longrightarrow 0$$

For a proof, see [12, (1.10.1)] or [16, (V.2.3)].

We have a natural transformation $$\begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "qC"'] \arrow[r, phantom, "\Downarrow i"] & \mathbf{Wald} \end{tikzcd}$$, where if $\mathcal{N} \in \mathbf{Exact}_{\mathbf{les}}$ then $i_{\mathcal{N}} : \mathcal{N} \hookrightarrow qC\mathcal{N}$ is the functor that concentrates an object in degree zero. When we take K-theory, we obtain the following

Theorem 1.7.12 (Gillet-Waldhausen). *There is a natural homotopy equivalence*

$$\begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "K"{name=U}] \arrow[r, bend right, "KqC"'{name=D}] & \mathbf{Spectra} \arrow[Rightarrow, from=U, to=D, "Ki"', "\simeq"] \end{tikzcd}$$

For a proof, see [12, (1.11.7)] or [16, (V.2.2)].

Theorem 1.7.13 (Waldhausen's approximation theorem). *Let $F : w\mathcal{C} \to v\mathcal{D}$ be an exact functor between Waldhausen categories such that $w\mathcal{C}$ satisfies the saturation axiom, has a cylinder functor and satisfies the cylinder axiom, and $v\mathcal{D}$ satisfies the saturation axiom. Suppose F satisfies the following conditions:*

1. *F reflects weak equivalences, i.e. if f is an arrow in $\mathcal{C}$ such that $Ff \in v$ then $f \in w$.*

2. *(Approximation property.) If $C \in \mathcal{C}, D \in \mathcal{D}$ and $f : FC \to D$, there exists a $C' \in \mathcal{C}$, a map $a : C \to C'$ and a weak equivalence $f' : FC' \to D$ such that the following diagram commutes:*

$$\begin{tikzcd} & FC' \arrow[d, dotted, "f'", "\simeq"'] \\ FC \arrow[ur, dotted, "Fa"] \arrow[r, "f"'] & D \end{tikzcd}$$

Then F induces a homotopy equivalence $KF : Kw\mathcal{C} \to Kv\mathcal{D}$.

For a proof, see [12, (1.9.1)], [14, (1.6.7)] or [16, (V.2.4)].

Chapter 2

The relative K-theory of $\Omega\mathcal{N}$

Let $\mathcal{N}$ be an exact category that supports long exact sequences.

Recall (remark 1.6.6.2) that there are three natural transformations given by Δ:

$$\begin{tikzcd} \mathbf{Exact} \arrow[r, bend left, "C"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow\Delta"] & \mathbf{Exact} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "qC"] \arrow[r, bend right, "qB"'] \arrow[r, phantom, "\Downarrow q\Delta"] & \mathbf{Wald} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "C^q"] \arrow[r, bend right, "B^q"'] \arrow[r, phantom, "\Downarrow\Delta^q"] & \mathbf{Exact}_{\mathbf{les}} \end{tikzcd} \tag{2.1}$$

The last one is the main tool that will allow us to describe the K-groups of $\mathcal{N}$ by generators and relations.

We recall observation 1.6.6.4. We will use it freely from now on.

Definition 2.0.14. We will denote by Ω the functor $\Delta^q : \mathbf{Exact}_{\mathbf{les}} \to \mathrm{Arr}\,(\mathbf{Exact}_{\mathbf{les}})$. More explicitely, $\Omega\mathcal{N}$ is the exact functor $C^q\mathcal{N} \xrightarrow{\Delta} B^q\mathcal{N}$. If $F : \mathcal{M} \to \mathcal{N}$ is an exact functor between exact categories, then ΩF is the following arrow in $\mathrm{Arr}\,(\mathbf{Exact}_{\mathbf{les}})$:

$$\begin{tikzcd} C^q\mathcal{M} \arrow[r, "\Delta"] \arrow[d, "C^qF"'] & B^q\mathcal{M} \arrow[d, "B^qF"] \\ C^q\mathcal{N} \arrow[r, "\Delta"'] & B^q\mathcal{N} \end{tikzcd}$$

We will also denote by Ω the inverse of the suspension isomorphism: $\Omega := \Sigma^{-1} : \mathbf{Spectra} \to \mathbf{Spectra}$. It passes to the homotopy category as an isomorphism too. Explicitely, if X is a spectrum then ΩX is the shifted spectrum $X[-1]$, and it is defined obviously on morphisms.

The relative K-theory spectrum $K[C^q\mathcal{N} \overset{\Delta}{\to} B^q\mathcal{N}]$, which we will write as $K\Omega\mathcal{N}$, fits into a homotopy fibration sequence as follows:

$$KC^q\mathcal{N} \xrightarrow{K\Delta} KB^q\mathcal{N} \longrightarrow K\Omega\mathcal{N}$$

If $F : \mathcal{M} \to \mathcal{N}$ is an arrow in $\mathbf{Exact}_{\mathbf{les}}$, we have an induced map $K\Omega F : K\Omega\mathcal{M} \to K\Omega\mathcal{N}$ in relative K-theory, making the following diagram commute:

$$\begin{tikzcd}
KC^q\mathcal{M} \arrow[r, "K\Delta"] \arrow[d, "KC^qF"'] & KB^q\mathcal{M} \arrow[r] \arrow[d, "KB^qF"] & K\Omega\mathcal{M} \arrow[d, "K\Omega F"] \\
KC^q\mathcal{N} \arrow[r, "K\Delta"] & KB^q\mathcal{N} \arrow[r] & K\Omega\mathcal{N}
\end{tikzcd}$$

The functor $K\Omega$ is what we aim to describe in this chapter. From this description we will be able to obtain a natural presentation of $K_1\mathcal{N}$ in the final chapter.

2.1 A stepping stone: $K\Omega \overset{\simeq}{\rightsquigarrow} KqB^b$

The first $\Delta : C\mathcal{N} \to B\mathcal{N}$ in (2.1) becomes a homotopy equivalence after passing to K-theory:

Lemma 2.1.1. *There is a natural homotopy equivalence* $\begin{tikzcd} \mathbf{Exact} \arrow[r, bend left, "KC"{name=U}] \arrow[r, bend right, "KB"'{name=D}] & \mathbf{Spectra} \arrow[Rightarrow, from=U, to=D, "K\Delta"] \end{tikzcd}$.

Proof. Since Δ is natural, $K\Delta$ is too; we need to check that $K\Delta : KC\mathcal{N} \to KB\mathcal{N}$ is a homotopy equivalence.

For $n > m$ we let $C\mathcal{N}_{[n,m]}$ (resp. $B\mathcal{N}_{[n,m]}$, $\mathrm{Gr}\mathcal{N}_{[n,m]}$) denote the full exact subcategory of $C\mathcal{N}$ (resp. $B\mathcal{N}$, $\mathrm{Gr}\mathcal{N}$) whose objects are the chain complexes (resp. binary chain complexes, graded objects) supported in $[n, m]$. The diagonal functor $\Delta : C\mathcal{N} \to B\mathcal{N}$ restricts to functors $\Delta : C\mathcal{N}_{[n,m]} \to B\mathcal{N}_{[n,m]}$.

There are "forget the differential(s)" functors $C\mathcal{N}_{[n,m]} \to \mathrm{Gr}\mathcal{N}_{[n,m]}$ and $B\mathcal{N}_{[n,m]} \to \mathrm{Gr}\mathcal{N}_{[n,m]}$ making the following square commute:

$$\begin{tikzcd}
C\mathcal{N}_{[n,m]} \arrow[r, "\Delta"] \arrow[d] & B\mathcal{N}_{[n,m]} \arrow[d] \\
\mathrm{Gr}\mathcal{N}_{[n,m]} \arrow[r, "\mathrm{id}"'] & \mathrm{Gr}\mathcal{N}_{[n,m]}
\end{tikzcd} \tag{2.2}$$

To prove that $K\Delta : KC\mathcal{N}_{[n,m]} \to KB\mathcal{N}_{[n,m]}$ is a homotopy equivalence, it suffices to see that the vertical functors in the previous diagram induce homotopy equivalences in K-theory. We postpone the proof of this to lemma 2.1.2 below, and we now concern ourselves with the general case.

We have inclusions $C\mathcal{N}_{[n,-n]} \to C\mathcal{N}_{[n+1,-n-1]}$ (resp. $\mathrm{Gr}\mathcal{N}_{[n,-n]} \to \mathrm{Gr}\mathcal{N}_{[n+1,-n-1]}$) for $n > 0$. Taking colimits in the diagrams (2.2) for these intervals, we obtain the following commutative square:

$$\begin{tikzcd}
C\mathcal{N} \arrow[r, "\Delta"] \arrow[d] & B\mathcal{N} \arrow[d] \\
\mathrm{Gr}\mathcal{N} \arrow[r, "\mathrm{id}"'] & \mathrm{Gr}\mathcal{N}
\end{tikzcd}$$

Thus what we have just seen combined with compatibility of K-theory with filtered colimits (proposition 1.7.5) and the fact that the colimit of homotopy equivalences is a homotopy equivalence prove the lemma. ☐

To complete the above proof, we prove the following lemma, which is exercise [16, (V.1.5)].

Lemma 2.1.2. *The “forget the differential(s)” functors $C\mathcal{N}_{[n,m]} \to \mathrm{Gr}\mathcal{N}_{[n,m]}$ and $B\mathcal{N}_{[n,m]} \to \mathrm{Gr}\mathcal{N}_{[n,m]}$ introduced in the proof of lemma 2.1.1 induce homotopy equivalences of K-theory spectra, for any $n \geq m$.*

Proof. We will prove it for chain complexes. The result for binary chain complexes is proven analogously.

We proceed by induction on $n - m$. If $n - m = 0$ the result is obvious. For the general case we will use the additivity theorem.

Let us introduce some notation: $\mathcal{A} = C\mathcal{N}_{[m,m]}$, $\mathcal{B} = C\mathcal{N}_{[n,m]}$ and $\mathcal{C} = C\mathcal{N}_{[n,m+1]}$. Denote by i and j the inclusion functors $\mathcal{A} \to \mathcal{B}$, $\mathcal{C} \to \mathcal{B}$ respectively.

We define functors $F' : \mathcal{B} \to \mathcal{A}$ and $F'' : \mathcal{B} \to \mathcal{C}$ truncating the complexes, as follows. For a chain complex $C_n \to \cdots \to C_m$, the functor F' maps it to C_m and the functor F'' maps it to $C_n \to \cdots \to C_{m+1}$. They are defined obviously on arrows. We have a short exact sequence of functors $0 \to iF' \to \mathrm{id}_{\mathcal{B}} \to jF'' \to 0$, where the first natural transformation is given by inclusion of chain complexes and the second one by quotienting:

$$\begin{tikzcd}
0 \arrow[r] & \ldots \arrow[r] \arrow[d] & 0 \arrow[r] & C_m \\
C_n \arrow[r] & \ldots \arrow[r] \arrow[d] & C_{m+1} \arrow[r] & C_m \\
C_n \arrow[r] & \ldots \arrow[r] & C_{m+1} \arrow[r] & 0
\end{tikzcd}$$

The additivity theorem 1.7.7 gives a homotopy

$$KiF' + KjF'' \simeq \mathrm{id}_{K\mathcal{B}} \tag{2.3}$$

Consider the composition $\mathcal{A} \times \mathcal{C} \to \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ where the first functor is $i \times j$ and the second functor is the coproduct. Denote the composite functor by π.

Consider the composition $\mathcal{B} \to \mathcal{B} \times \mathcal{B} \to \mathcal{A} \times \mathcal{C}$, where the first functor is the obvious diagonal functor and the second one is the product $F' \times F''$. Denote the composite functor by φ.

We have therefore two functors $\mathcal{A} \times \mathcal{C} \underset{\varphi}{\overset{\pi}{\rightleftarrows}} \mathcal{B}$ satisfying $\varphi \circ \pi = \mathrm{id}$, therefore $K\varphi \circ K\pi = \mathrm{id}_{K\mathcal{A} \times K\mathcal{C}}$, recalling that $K(\mathcal{A} \times \mathcal{C}) \cong K\mathcal{A} \times K\mathcal{C}$ (proposition 1.7.5).

Observe that $\pi \circ \varphi = iF' \times jF''$. Passing to K-theory we get that $K\pi \circ K\varphi = K(iF' \times jF'') = KiF' + KjF''$. Indeed, the H-space structure of $K\mathcal{B}$ is induced by the product: see the proof of [16, (V.1.2)] or [14, (1.3.3)].

Combining this with (2.3) we get that $K\pi$ and $K\varphi$ are mutually inverse homotopy equivalences. Recalling the definitions of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, the induction hypothesis and the base case combine to give that the following composition, which is the "forget the differential" functor, yields a homotopy equivalence in K-theory:

$$C\mathcal{N}_{[n,m]} \xrightarrow{\varphi} C\mathcal{N}_{[m,m]} \times C\mathcal{N}_{[n,m+1]} \longrightarrow \mathrm{Gr}\mathcal{N}_{[m,m]} \times \mathrm{Gr}\mathcal{N}_{[n,m+1]} \xrightarrow{\cong} \mathrm{Gr}\mathcal{N}_{[n,m]}$$

This finishes the proof. □

We will now see that there is a homotopy equivalence from the relative K-theory of the last Δ in (2.1) (which is Ω) to the loop spectrum of the relative K-theory of the second one.

Theorem 2.1.3. *There is a natural homotopy equivalence* $\begin{tikzcd}\mathbf{Exact_{les}} \arrow[r, bend left, "K\Omega"] \arrow[r, bend right, "\Omega K q\Delta"'] \arrow[r, phantom, "\Downarrow\simeq"] & \mathbf{Spectra}\end{tikzcd}$.

The functor in the bottom is the following composition:

$$\mathbf{Exact_{les}} \xrightarrow{q\Delta} \mathrm{Arr}\,(\mathbf{Wald}) \xrightarrow{K} \mathbf{Spectra} \xrightarrow{\Omega} \mathbf{Spectra}$$

Proof. Let us construct a homotopy equivalence $K\Omega\mathcal{N} \xrightarrow{\simeq} \Omega K[qC\mathcal{N} \overset{\Delta}{\to} qB\mathcal{N}]$.

We apply Waldhausen's localization theorem (1.7.10) to the categories of weak equivalences $i \subset q$ in $C\mathcal{N}$ and in $B\mathcal{N}$, thus yielding two homotopy fibration sequences:

$$KC^q\mathcal{N} \longrightarrow KC\mathcal{N} \longrightarrow KqC\mathcal{N}$$

$$KB^q\mathcal{N} \longrightarrow KB\mathcal{N} \longrightarrow KqB\mathcal{N}$$

There are maps $K\Delta$ from the first row to the second row making the ladder diagram commute, since it obviously commutes at the categorical level, i.e. before taking K-theory spectra. We complete them to the homotopy fibration sequences of relative K-theory, and we take the induced maps:

$$\begin{tikzcd}
KC^q\mathcal{N} \arrow[r] \arrow[d, "K\Delta"'] & KC\mathcal{N} \arrow[r] \arrow[d, "K\Delta"'] & KqC\mathcal{N} \arrow[d, "K\Delta"'] \\
KB^q\mathcal{N} \arrow[r] \arrow[d] & KB\mathcal{N} \arrow[r] \arrow[d] & KqB\mathcal{N} \arrow[d] \\
K\Omega\mathcal{N} \arrow[r] & K[C\mathcal{N} \overset{\Delta}{\to} B\mathcal{N}] \arrow[r] & K[qC\mathcal{N} \overset{\Delta}{\to} qB\mathcal{N}]
\end{tikzcd} \tag{2.4}$$

Lemma 2.1.1 says that the $K\Delta$ in the middle is a homotopy equivalence. Thus the cofiber of this $K\Delta$ is contractible. Therefore the map $K[C\mathcal{N} \overset{\Delta}{\to} B\mathcal{N}] \to K[qC\mathcal{N} \overset{\Delta}{\to} qB\mathcal{N}]$ is homotopic to a constant map. Thus its homotopy fiber is $\Omega K[qC\mathcal{N} \overset{\Delta}{\to} qB\mathcal{N}]$.

Since the last line of (2.4) is a homotopy fibration sequence, we get the desired homotopy equivalence $K\Omega\mathcal{N} \to \Omega K[qC\mathcal{N} \overset{\Delta}{\to} qB\mathcal{N}]$.

We now check naturality. Let $F : \mathcal{M} \to \mathcal{N}$ be an arrow in $\mathbf{Exact_{les}}$. Naturality follows from naturality of inclusions, of the three Δ maps, of relative K-theory and of homotopy fibers.

We have the following diagram, which is commutative except for the square involving the dashed arrows, which is homotopy commutative and thus the proof is finished (this is in agreement with definition 1.7.2):

$$\begin{tikzcd}
 & KC^q\mathcal{M} \arrow[rr] \arrow[dl] \arrow[dd] & & KC\mathcal{M} \arrow[rr] \arrow[dl] \arrow[dd] & & KqC\mathcal{M} \arrow[dl] \arrow[dd] \\
KC^q\mathcal{N} \arrow[rr] \arrow[dd] & & KC\mathcal{N} \arrow[rr] \arrow[dd] & & KqC\mathcal{N} \arrow[dd] & \\
 & KB^q\mathcal{M} \arrow[rr] \arrow[dl] \arrow[dd] & & KB\mathcal{M} \arrow[rr] \arrow[dl] \arrow[dd] & & KqB\mathcal{M} \arrow[dl] \arrow[dd] \\
KB^q\mathcal{N} \arrow[rr] \arrow[dd] & & KB\mathcal{N} \arrow[rr] \arrow[dd] & & KqB\mathcal{N} \arrow[dd] & \\
 & K\Omega\mathcal{M} \arrow[rr] \arrow[dl, "K\Omega F"'] \arrow[dd, dotted, "\simeq"'] & & K\Delta\mathcal{M} \arrow[rr, "\simeq *"] \arrow[dl] & & Kq\Delta\mathcal{M} \arrow[dl] \\
K\Omega\mathcal{N} \arrow[rr] \arrow[dd, dotted, "\simeq"'] & & K\Delta\mathcal{N} \arrow[rr, "\simeq *"'] & & Kq\Delta\mathcal{N} & \\
 & \Omega Kq\Delta\mathcal{M} \arrow[dl, "\Omega Kq\Delta F"'] \arrow[uurr, bend right] & & & & \\
\Omega Kq\Delta\mathcal{N} \arrow[uurr, bend right] & & & & &
\end{tikzcd}$$

□

Corollary 2.1.4. *There is a natural homotopy equivalence*
$$\begin{tikzcd}
\mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "K\Omega", ""{name=U, below}] \arrow[r, bend right, "\Omega^2 Kq\perp"', ""{name=D}] & \mathbf{Spectra} \arrow[from=U, to=D, Rightarrow, "\simeq"]
\end{tikzcd}.$$

Proof. We have the following commutative diagram, where the two vertical and horizontal sequences are homotopy fibrations:

$$\begin{tikzcd}
KqC\mathcal{N} \arrow[r, "K\Delta"] \arrow[d, "\mathrm{id}"'] & KqB\mathcal{N} \arrow[r] \arrow[d, "K\perp"] & K[qC\mathcal{N} \overset{\Delta}{\to} qB\mathcal{N}] \\
KqC\mathcal{N} \arrow[r, "\mathrm{id}"'] & KqC\mathcal{N} \arrow[d] & \\
 & K[qB\mathcal{N} \overset{\perp}{\to} qC\mathcal{N}] &
\end{tikzcd}$$

Taking homotopy fibers and their induced maps, we obtain the following commutative diagram, where the displayed homotopy equivalence is theorem 2.1.3:

$$\begin{tikzcd}
 & * \arrow[r] \arrow[d] & \Omega K[qB\mathcal{N} \overset{\perp}{\to} qC\mathcal{N}] \arrow[d] \\
K\Omega\mathcal{N} \overset{\simeq}{\to} \Omega K[qC\mathcal{N} \overset{\perp}{\to} qB\mathcal{N}] \arrow[r] \arrow[d] & KqC\mathcal{N} \arrow[r, "K\Delta"] \arrow[d, "\mathrm{id}"'] & KqB\mathcal{N} \arrow[d, "K\perp"] \\
* \arrow[r] & KqC\mathcal{N} \arrow[r, "\mathrm{id}"'] & KqC\mathcal{N}
\end{tikzcd}$$

Since iterated homotopy fibers are homotopy equivalent (theorem [11, (8.57)]), we get the result by comparing the homotopy fiber of the vertical bottom left map with the one of the horizontal top right map.

Naturality is proven in a similar way as in theorem 2.1.3. It follows from naturality of homotopy fibers, naturality of the homotopy equivalence in said theorem, and naturality of the homotopy equivalence between iterated homotopy fibers. □

Theorem 2.1.5. *There is a natural homotopy equivalence* $\begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "KbB"] \arrow[r, bend right, "KqC"'] \arrow[r, phantom, "\simeq \Downarrow Kb\perp"] & \mathbf{Spectra} \end{tikzcd}$.

Proof. We already observed naturality in remark 1.6.6.3. Let us now check that the map $K\perp : KbB\mathcal{N} \to KqC\mathcal{N}$ is a homotopy equivalence.

Let $F : qC\mathcal{N} \to bB\mathcal{N}$ be the exact functor defined as $(N, d) \mapsto (N, 0, d)$ on objects and obviously on arrows. We will check that KF is a homotopy inverse to $K\perp$: since $\perp \circ F = \mathrm{id}$, we need only check that $KF \circ K\perp \simeq \mathrm{id}$. To prove this, we will check that both $KF \circ K\perp$ and id are homotopic to KG where $G : bB\mathcal{N} \to bB\mathcal{N}$ is defined through a binary mapping cylinder construction:

$$(N, d, d') \xmapsto{G} \mathbb{T}((N, d) \overset{0}{\to} (N, d), (N, d') \overset{\mathrm{id}}{\to} (N, d'))$$

By virtue of proposition 1.7.9, it suffices to see that there are weak equivalences from $F \circ \perp$ to G and from id to G.

These weak equivalences are induced by the front and back inclusions. Let us check this. First observe that $\perp G(N, d, d') = \mathbb{T}(\mathrm{id}_{(N,d')})$.

The front inclusion $\lambda : (N, d') \to \perp G(N, d, d')$ induces a map $\tilde{\lambda} : (N, d, d') \to G(N, d, d')$ defined in the same way. As $\mathrm{id}_{(N,d')}$ is a quasi-isomorphism, then λ is one too (remark 1.2.6), and thus $\tilde{\lambda}$ is in b.

On the other hand, the back inclusion $\rho : (N, d') \to \perp G(N, d, d')$ induces $\tilde{\rho} : (F \circ \perp)(N, d, d') \to G(N, d, d')$ defined in the same way. But ρ is a quasi-isomorphism by the cylinder axiom, and thus $\tilde{\rho}$ is in b.

In summary, we have the following weak equivalences in $bB\mathcal{N}$:

$$(N, d, d') \xrightarrow[\simeq]{\tilde{\lambda}} G(N, d, d') \xleftarrow[\simeq]{\tilde{\rho}} (F \circ \perp)(N, d, d')$$

These arrows are the components of natural transformations (this is part of the definition of cylinder functors) which are then weak equivalences:

$$\mathrm{id}_{bB\mathcal{N}} \overset{\simeq}{\Longrightarrow} G \overset{\simeq}{\Longleftarrow} F \circ \perp$$

Proposition 1.7.9 applies, finishing the proof. □

Corollary 2.1.6. *There is a natural zigzag homotopy equivalence* $\begin{tikzcd}\mathbf{Exact_{les}} \arrow[r, bend left, "K\Omega"] \arrow[r, bend right, "\Omega KqB^b"'] \arrow[r, phantom, "\wr\simeq"] & \mathbf{Spectra}\end{tikzcd}$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
KqB^b\mathcal{N} \arrow[r] \arrow[d, dotted, "\simeq"'] & KqB\mathcal{N} \arrow[r, "Ki"] \arrow[d, "\mathrm{id}"'] & KbB\mathcal{N} \arrow[d, "\simeq"', "K\perp"] & \\
\Omega K[qB\mathcal{N} \overset{\perp}{\to} qC\mathcal{N}] \arrow[r] & KqB\mathcal{N} \arrow[r, "K\perp"] & KqC\mathcal{N} \arrow[r] & K[qB\mathcal{N} \overset{\perp}{\to} qC\mathcal{N}]
\end{tikzcd}$$

where $i : qB\mathcal{N} \to bB\mathcal{N}$ is the inclusion and the right vertical map is a homotopy equivalence thanks to theorem 2.1.5. The bottom sequence is a homotopy fibration sequence, and the top one is one too thanks to Waldhausen's localization theorem 1.7.10. The right square being commutative, there is an induced dotted map making the left square commute up to homotopy. This map is a homotopy equivalence since id and $K\perp$ are.

Thus there is a homotopy equivalence $KqB^b\mathcal{N} \overset{\simeq}{\to} \Omega K[qB\mathcal{N} \overset{\perp}{\to} qC\mathcal{N}]$. It defines a natural homotopy equivalence. Indeed, naturality follows from naturality of the induced dotted map, of $\perp$ and of inclusions. Applying the loop functor and corollary 2.1.4 yields:

$$K\Omega\mathcal{N} \xrightarrow{\simeq} \Omega^2 K[qB\mathcal{N} \overset{\perp}{\to} qC\mathcal{N}] \xleftarrow{\simeq} \Omega KqB^b\mathcal{N}$$

Since both of these homotopy equivalences are natural, we are done. □

2.2 The main result: $V^0\Omega K \overset{\simeq}{\leftrightsquigarrow} K\Omega$

2.2.1 Whitehead and Postnikov towers

Definition 2.2.1. Let X be a spectrum and $t \in \mathbb{Z}$.

We denote by V_tX the stage t of the *Postnikov tower* of X. It is a spectrum satisfying $\pi_i(V_tX) = 0$ if $i > t$ and there is a map $i_X : X \to V_tX$ that induces an isomorphism on homotopy groups up to level t.

We denote by V^tX the stage t of the *Whitehead tower* of X. It is a spectrum satisfying $\pi_i(V^tX) = 0$ if $i < t$ and there is a map $p_X : V^tX \to X$ that induces an isomorphism on homotopy groups from level t onwards.

Both of these constructions are natural: we have functors and natural transformations

$$\begin{tikzcd}
\mathbf{Spectra} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "V_t"'] \arrow[r, phantom, "\Downarrow i"] & \mathbf{Spectra}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Spectra} \arrow[r, bend left, "V^t"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow p"] & \mathbf{Spectra}
\end{tikzcd}$$

The functors V_t and V^t respect homotopies.

The spectrum V^tX is called the *t-connected covering of* X, and V^0X is called the *connective part of* X.

Remark 2.2.2. Let us recall how these constructions are obtained, at least for pointed CW-complexes.

If $t \in \mathbb{N}$, V_tX is obtained by attaching cells of dimension $\geq t+2$ to X. See [13, (8.6.6)] for more details.

The space V^tX is obtained as the homotopy fiber of the inclusion $X \to V_{t+1}X$. This follows from the long exact homotopy sequence associated to the fibration sequence $\Omega V_{t+1}X \to V^tX \to X$.

Proposition 2.2.3. *For any $t \in \mathbb{Z}$ there are natural homotopy equivalences*

$$\begin{tikzcd} \mathbf{Spectra} \arrow[r, bend left, "V_{t-1}\Omega"{name=U}] \arrow[r, bend right, "\Omega V_t"'{name=D}] & \mathbf{Spectra} \arrow[Rightarrow, from=U, to=D, "\simeq"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Spectra} \arrow[r, bend left, "V^t\Omega"{name=U}] \arrow[r, bend right, "\Omega V^{t+1}"'{name=D}] & \mathbf{Spectra} \arrow[Rightarrow, from=U, to=D, "\simeq"] \end{tikzcd}$$

Proof. We will do the proof for pointed CW-complexes.

Since $V_{t-1}\Omega X$ is obtained from ΩX by attaching cells of dimension $\geq t+1$ and since $\pi_k(\Omega V_tX) = 0$ if $k \geq t$, the inclusion $i_{\Omega X} : \Omega X \to V_{t-1}\Omega X$ induces a natural bijection of sets of pointed homotopy classes of maps (this is the content of [13, (8.6.5)]):

$$i^*_{\Omega X} : [V_{t-1}\Omega X, \Omega V_tX] \to [\Omega X, \Omega V_tX] \tag{2.5}$$

Now, Ωi_X lives on the right hand side of (2.5). Denote by $s_X : V_{t-1}\Omega X \to \Omega V_tX$ the corresponding map on the left hand side. It is the desired homotopy equivalence. Indeed, $i_{\Omega X}$ induces isomorphisms on homotopy groups up to dimension $t-1$ and Ωi_X does too. Seeing as both source and target of s_X have trivial homotopy groups from t onwards, this proves that s_X is a weak equivalence and thus a homotopy equivalence by Whitehead's theorem.

As for naturality, it follows from naturality of the bijection (2.5) and of i that s is also a natural transformation.

We have the following commutative diagram where the vertical and horizontal sequences are homotopy fibrations:

$$\begin{tikzcd} & \Omega V^{t+1}X \arrow[d] & \\ V^t\Omega X \arrow[r] & \Omega X \arrow[r, "i_{\Omega X}"] \arrow[d, "\Omega i_X"'] & V_{t-1}\Omega X \arrow[dl, "s_X"] \\ & \Omega V_tX & \end{tikzcd}$$

Passing to homotopy fibers yields a map that is also a homotopy equivalence:

$$\begin{tikzcd} V^t\Omega X \arrow[r] \arrow[d, dotted, "\simeq"] & \Omega X \arrow[r, "i_{\Omega X}"] \arrow[d, "\mathrm{id}"] & V_{t-1}\Omega X \arrow[d, "s_X"] \\ \Omega V^{t+1}X \arrow[r] & \Omega X \arrow[r, "\Omega i_X"'] & \Omega V^tX \end{tikzcd}$$

Naturality follows from naturality of i, s, and of homotopy fibers. □

We aim to prove that there is a natural zigzag homotopy equivalence $K\Omega \overset{\simeq}{\rightsquigarrow} V^0\Omega K$. The previous proposition suggests that we might as well start by investigating V^1K first, which is what we will do.

2.2.2 Description of V^1K

In this section we will freely use the explicit description of the K_0 group of a Waldhausen category (proposition 1.7.4).

Definition 2.2.4. We define the *Euler characteristic* of a graded object N of $\mathcal{N}$:

$$\chi(N) := \sum_i (-1)^i [N_i] \in K_0\mathcal{N}$$

The Euler characteristic of a chain complex is by definition the Euler characteristic of its underlying graded object.

Proposition 2.2.5 (Additivity of χ)**.** *If* $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ *is an exact sequence of graded objects, then* $\chi(N) = \chi(N') + \chi(N'')$. *In particular,* $\chi(N' \oplus N'') = \chi(N') + \chi(N'')$ *for any* N', N'' *graded objects.*

Proof. For every $i \in \mathbb{Z}$ we have a short exact sequence in $\mathcal{N}$:

$$0 \longrightarrow N_i' \longrightarrow N_i \longrightarrow N_i'' \longrightarrow 0$$

Thus $[N_i] = [N_i'] + [N_i'']$ in $K_0\mathcal{N}$. Hence summing over all i while alternating signs proves the assertion. □

Proposition 2.2.6. *If* (N, d) *is an acyclic chain complex in* $\mathcal{N}$, *then* $\chi(N) = 0$.

Proof. Suppose N is supported on the interval $[n, m]$. We have the following diagram with short exact sequences in the shape of the symbol "$\wedge$":

$$\begin{tikzcd}
0 \arrow[rr] & & N_n \arrow[rr, "d_n"] \arrow[dr] & & N_{n-1} \arrow[rr, "d_{n-1}"] \arrow[dr] & & N_{n-2} \arrow[r] & \cdots \\
& 0 \arrow[ur] & & Z_{n-1} \arrow[ur] \arrow[dr] & & Z_{n-2} \arrow[ur] \arrow[dr] & & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

By proposition 2.2.5, we have $[N_i] = [Z_i] + [Z_{i-1}]$ for all $i \in \mathbb{Z}$. Combining this with the fact that $Z_i = 0$ for $i \geq n$, $Z_m = N_m$ and $Z_i = 0$ for $i < m$ gives the result. □

Lemma 2.2.7. *If* N *is a graded object, then* $\chi(N[i]) = (-1)^i \chi(N)$ *for every* $i \in \mathbb{Z}$.

Proof. We prove it by induction. The result is obvious for $i \in \{1, 0, -1\}$, and the general case follows since $N[i] = N[i-1][1]$ if $i > 1$ and similarly if $i < 1$. □

Proposition 2.2.8. *If there is a quasi-isomorphism of chain complexes $N \to M$, then $\chi(N) = \chi(M)$.*

Proof. Let $f : N \to M$ be a quasi-isomorphism in $C\mathcal{N}$, that is, $\mathrm{cone}(f)$ is acyclic. Recall the short exact sequence in $C\mathcal{N}$ from remark 1.5.2.1:

$$0 \longrightarrow M \longrightarrow \mathrm{cone}(f) \longrightarrow N[-1] \longrightarrow 0$$

By propositions 2.2.5, 2.2.6 and lemma 2.2.7, we have

$$0 = \chi(\mathrm{cone}(f)) = \chi(M) + \chi(N[-1]) = \chi(M) - \chi(N) \qquad \square$$

Proposition 2.2.9. *The Euler characteristic defines a natural transformation*

$$\begin{tikzcd} \mathbf{Exact_{les}} \arrow[r, bend left, "K_0qC"] \arrow[r, bend right, "K_0"'] \arrow[r, phantom, "\Downarrow\chi"] & \mathbf{Ab} \end{tikzcd}$$

such that $\chi_{\mathcal{N}}$ is an abelian group epimorphism for every $\mathcal{N} \in \mathbf{Exact_{les}}$. [1]

Proof. Propositions 2.2.5 and 2.2.8 give a well defined abelian group homomorphism $\chi_{\mathcal{N}} : K_0qC\mathcal{N} \to K_0\mathcal{N}$. It is surjective, since it has a right inverse induced by the inclusion $\mathcal{N} \hookrightarrow qC\mathcal{N}$ defined by concentrating an object in degree zero.

We check naturality: if $F : \mathcal{M} \to \mathcal{N}$ is an arrow in $\mathbf{Exact_{les}}$ and $[M] \in K_0qC\mathcal{M}$, then

$$\chi_{\mathcal{N}}([CF(M)]) = \sum_i (-1)^i [FM_i] = K_0F\left(\sum_i (-1)^i [M_i]\right) = K_0F(\chi_{\mathcal{M}}([M])) \qquad \square$$

We will apply Thomason's cofinality theorem 1.7.11 to $\chi_{\mathcal{N}}$. The Waldhausen category with a cylinder functor we will consider in this application is $qC\mathcal{N}$, and $q(C\mathcal{N})_{\chi}$ is by definition the full Waldhausen subcategory of $qC\mathcal{N}$ whose objects are the chain complexes with zero Euler characteristic, with the quasi-isomorphisms as weak equivalences; we will denote it by $qC_x\mathcal{N}$.

Observe that qC_x defines a functor $qC_x : \mathbf{Exact_{les}} \to \mathbf{Wald}$. Indeed, this is true since by naturality of χ (proposition 2.2.9), if $F : \mathcal{M} \to \mathcal{N}$ is an exact functor, then $qCF : qC\mathcal{M} \to qC\mathcal{N}$ maps $qC_x\mathcal{M}$ into $qC_x\mathcal{N}$.

Corollary 2.2.10. *There is a natural homotopy equivalence*

$$\begin{tikzcd} \mathbf{Exact_{les}} \arrow[r, bend left, "KqC_x"] \arrow[r, bend right, "V^1K"'] \arrow[r, phantom, "\Downarrow\simeq"] & \mathbf{Spectra} \end{tikzcd}.$$

Proof. As we stated above, we apply Thomason's cofinality theorem 1.7.11 to the abelian group epimorphism $\chi_{\mathcal{N}} : K_0qC\mathcal{N} \to K_0\mathcal{N}$. We can naturally model "$K_0\mathcal{N}$" as $V_0K\mathcal{N}$ making diagram (2.6) commute. This diagram shows two homotopy fibration sequences: the one on top is the one coming from Thomason's theorem, and the one on the bottom is the one defining $V^1K\mathcal{N}$. The middle map is given by Gillet-Waldhausen's theorem 1.7.12.

$$\begin{tikzcd} KqC_x\mathcal{N} \arrow[r] \arrow[d, dotted, "\simeq"] & KqC\mathcal{N} \arrow[r] \arrow[d, "\simeq"] & V_0K\mathcal{N} \arrow[d, "\mathrm{id}"] \\ V^1K\mathcal{N} \arrow[r] & K\mathcal{N} \arrow[r] & V_0K\mathcal{N} \end{tikzcd} \tag{2.6}$$

---

[1] It can be proven that χ is in fact a natural isomorphism ([16, II.9.2.2]), but we won't be needing it.

There is therefore an induced homotopy equivalence $Kq C_x\mathcal{N} \to V^1 K\mathcal{N}$. It is natural. Indeed, the only map we don't know to be natural is the map $KqC\mathcal{N} \to V_0 K\mathcal{N}$. But this is the case, as can be seen from the proof of Thomason's theorem. To prove it one gets a homotopy fibration sequence coming from Waldhausen's localization theorem 1.7.10: here the second map is natural, as it is induced by an inclusion. In this homotopy fibration sequence, there is a homotopy equivalence from the last spectrum to "G", and this equivalence is also seen to be natural. We refer the reader to the proofs of theorems [16, (IV.8.10)] and [16, (V.2.3)]. □

2.2.3 Proof of the main result

In a footnote to proposition 2.2.9 we remarked that $\chi : K_0 qC\mathcal{N} \to K_0\mathcal{N}$ was an isomorphism. Of course, if we consider the function $\chi : \mathrm{Ob}(C\mathcal{N}) \to K_0\mathcal{N}$, it has a non-trivial kernel, in the sense that there are non-zero chain complexes with vanishing Euler characteristic, e.g. $0 \longrightarrow N_0 \xrightarrow{0} N_0 \longrightarrow 0$ if N_0 is not a zero object. We now seek to characterize such chain complexes. In fact, we can consider the function $\chi : \mathrm{Ob}(\mathrm{Gr}\mathcal{N}) \to K_0\mathcal{N}$, and it is its kernel (i.e. the preimage of $\{0\}$) that we will describe.

Definition 2.2.11. Let N be a graded object of $\mathcal{N}$. We say that N is

- *potentially acyclic* if there exists a differential d on N such that the chain complex (N, d) is acyclic,
- *stably potentially acyclic* if there exists a potentially acyclic graded object N' of $\mathcal{N}$ such that $N \oplus N'$ is potentially acyclic.

Of course, potentially acyclic objects are stably potentially acyclic.

Remark 2.2.12. (Stably) potentially acyclic graded objects have zero Euler characteristic, by virtue of proposition 2.2.6 and of additivity of χ.

Lemma 2.2.13. *A graded object N of $\mathcal{N}$ satisfies $\chi(N) = 0$ if and only if it is stably potentially acyclic.*

Proof. We will now reproduce and use a particular case of "Grayson's trick" ([2, (9)]).

We have a forgetful functor $C^q\mathcal{N} \to \mathrm{Gr}\mathcal{N}$ which associates to an acyclic chain complex its underlying graded object. We can consider equivalence in $\mathrm{Gr}\mathcal{N}$ modulo the image of this functor: explicitely, if $N, N' \in \mathrm{Gr}\mathcal{N}$ then we write $N \sim N'$ if there exist M, M' potentially acyclic graded objects such that $N \oplus M \cong N' \oplus M'$.

This is obviously an equivalence relation. We can consider its quotient set, which we will denote by G, and we will denote the equivalence class of $N \in \mathrm{Gr}\mathcal{N}$ by $\langle N \rangle$.

Claim: G is an abelian group with the binary operation defined as $\langle N \rangle + \langle N' \rangle := \langle N \oplus N' \rangle$. The class $\langle 0 \rangle$ is the identity element, and $\langle N \rangle = \langle 0 \rangle$ if and only if N is stably potentially acyclic. The opposite of $\langle N \rangle$ is $\langle N[-1] \rangle$.

Proof: The only non-trivial assertion is the description of the opposite of $\langle N \rangle$. It suffices to observe that $N \oplus N[-1]$ is potentially acyclic: define $d : N_n \oplus N_{n-1} \to N_{n-1} \oplus N_{n-2}$ as $d(n, m) = (m, 0)$; it endows $N \oplus N[-1]$ with an acyclic differential. Thus

$$\langle N \rangle + \langle N[-1] \rangle = \langle N \oplus N[-1] \rangle = \langle 0 \rangle \qquad \blacksquare$$

We have a well defined Euler characteristic homomorphism $\chi : G \to K_0\mathcal{N}$. To see this, let $N \sim N'$, i.e. there exist M, M' potentially acyclic such that $N \oplus M \cong N' \oplus M'$. Since χ is additive, invariant under (quasi)-isomorphism and zero on acyclic chain complexes (propositions 2.2.5, 2.2.8 and observation 2.2.12), we have

$$\chi(N) = \chi(N) + \chi(M) = \chi(N \oplus M) = \chi(N' \oplus M') = \chi(N') + \chi(M') = \chi(N')$$

Now we must prove that χ is an isomorphism, for if that's the case, then $\chi(\langle N \rangle) = \chi(N) = 0$ if and only if $\langle N \rangle = \langle 0 \rangle$, if and only if N is stably potentially acyclic, proving the lemma.

We define a group homomorphism $j : K_0\mathcal{N} \to G$. If $N \in \mathcal{N}$, write $N_\bullet$ for the graded object of $\mathcal{N}$ that is concentrated by N in degree 0. Now define $j([N]) = \langle N_\bullet \rangle$. We must check that this is well defined. It is immediate by definition that $N \cong N'$ implies $\langle N \rangle = \langle N' \rangle$.

Now we make the observation that if N is a graded object of $\mathcal{N}$, then $\langle N[i] \rangle = (-1)^i \langle N \rangle$ for all $i \in \mathbb{Z}$. We have already checked it for $i = -1$. That proof is immediately adapted to see that it is true for $i = 1$. The general case follows by induction. So if $0 \longrightarrow N' \xrightarrow{i} N \xrightarrow{p} N'' \longrightarrow 0$ is a short exact sequence in $\mathcal{N}$, then

$$\langle N'_\bullet \rangle + \langle N''_\bullet \rangle - \langle N_\bullet \rangle = \langle N'_\bullet \rangle + \langle N''_\bullet[2] \rangle + \langle N_\bullet[1] \rangle = \langle N'_\bullet \oplus N''_\bullet[2] \oplus N_\bullet[1] \rangle = \langle 0 \rangle$$

the last equality being true because the graded object $N'_\bullet \oplus N''_\bullet[2] \oplus N_\bullet[1]$ can be endowed with i and p as an acyclic differential, hence it is potentially acyclic.

We need only check that j and χ are mutually inverse.

$$\chi j([N]) = \chi(\langle N_\bullet \rangle) = \chi(N_\bullet) = [N]$$

$$j\chi(\langle N \rangle) = j\left(\sum_i (-1)^i [N_i]\right) = \sum_i (-1)^i \langle N_i \rangle = \sum_i \langle N_i[i] \rangle = \left\langle \bigoplus_i N_i[i] \right\rangle = \langle N \rangle$$

thus finishing the proof. □

If $(N, d, d') \in B^b\mathcal{N}$, then $(N, d) \in C_x\mathcal{N}$. Indeed, as (N, d') is acyclic, $\chi(N) = 0$.

Thus we have an exact functor $\top : tB^b\mathcal{N} \to qC_x\mathcal{N}$, which defines a natural transformation

$$\begin{tikzcd} \mathbf{Exact_{les}} \arrow[r, bend left, "tB^b"{name=U}] \arrow[r, bend right, "qC_x"'{name=D}] & \mathbf{Exact_{les}} \arrow[from=U, to=D, phantom, "\Downarrow \top"] \end{tikzcd}$$

. When we take K-theory, we obtain the following

Theorem 2.2.14. *There is a natural homotopy equivalence*

$$\begin{tikzcd} \mathbf{Exact_{les}} \arrow[r, bend left, "KtB^b"{name=U}] \arrow[r, bend right, "KqC_x"'{name=D}] & \mathbf{Spectra} \arrow[from=U, to=D, phantom, "K\top \Downarrow \simeq"] \end{tikzcd}$$

.

Proof. To prove the theorem we will apply Waldhausen's approximation theorem 1.7.13 to the functor $\top : tB^b\mathcal{N} \to qC_x\mathcal{N}$. The conclusion of the approximation theorem gives exactly what we want.

We have already observed that $tB\mathcal{N}$ satisfies the saturation and cylinder axiom, and $qC_x\mathcal{N}$ satisfies the saturation axiom since $qC\mathcal{N}$ does. Thus the hypotheses of the approximation theorem are satisfied.

We have to check that $\top : tB^b\mathcal{N} \to qC_x\mathcal{N}$ satisfies two conditions. The fact that it reflects quasi-isomorphisms is merely the definition of t.

Let $N \in B^b\mathcal{N}$, $P \in C_x\mathcal{N}$ and $r : \top N \to P$. We have to see that there exist $M \in B^b\mathcal{N}$, $s : N \to M$ and a quasi-isomorphism $u : \top M \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \top M \arrow[d, dotted, "u", "\simeq"'] \\
\top N \arrow[ur, dotted, "\top s"] \arrow[r, "r"'] & P
\end{tikzcd} \tag{2.7}$$

Since $\chi(P) = 0$, by lemma 2.2.13 there exists a potentially acyclic $Y \in \mathrm{Gr}\mathcal{N}$ such that $\mathrm{gr}P \oplus Y$ has an acyclic differential z. Let y be an acyclic differential on Y. Let $i_1 : P \to P \oplus (Y, y)$ be the inclusion and define $r' = i_1 \circ r$:

$$\begin{tikzcd}
\top N \arrow[d, "r"'] \arrow[dr, "r'"] & \\
P \arrow[r, "i_1"'] & P \oplus (Y, y)
\end{tikzcd}$$

Let $Q = (\mathrm{gr}P \oplus Y, d_P \oplus y, z)$ and consider

$$r' : \top N \to \top Q = P \oplus (Y, y) \qquad 0 : \bot N \to \bot Q = (\mathrm{gr}P \oplus Y, z)$$

Define M as the binary mapping cylinder $M := \mathbb{T}(r', 0)$. Define $s : N \to M$ as the front inclusion.

Let π be the projection $\top M = \mathbb{T}(r') \to P \oplus (Y, y)$. Let $p_1 : P \oplus (Y, y) \to P$ be the projection, and define $u := p_1 \circ \pi$. We have the following commutative diagram:

$$\begin{tikzcd}
\top N \arrow[r, "\top s"] \arrow[dr, "r'"'] & \top M = \mathbb{T}(r') \arrow[d, "\pi"] \arrow[dr, "u"] & \\
 & P \oplus (Y, y) \arrow[r, "p_1"'] & P
\end{tikzcd}$$

We now check that M, s and u satisfy what we need:

- $M \in B^b\mathcal{N}$. The back inclusion $\rho : (\mathrm{gr}P \oplus Y, z) \to \mathbb{T}(\ \bot N \xrightarrow{0} (\mathrm{gr}P \oplus Y, z)\ ) = \bot M$ is a quasi-isomorphism (proposition 1.5.7) and $(\mathrm{gr}P \oplus Y, z)$ is acyclic, hence $\bot M$ is acyclic, i.e. $M \in B^b\mathcal{N}$ (remark 1.2.8).
- $u = p_1 \circ \pi$ is a quasi-isomorphism. Since (Y, y) is acyclic then p_1 is the identity in homology groups, hence it is a quasi-isomorphism. The morphism π is also a quasi-isomorphism, being the projection of a cylinder in the category $qC\mathcal{N}$ that satisfies the cylinder axiom. Thus their composition u is a quasi-isomorphism.
- The commutativity of (2.7) is read in the commutativity of the following diagram:

$$\begin{tikzcd}
\top N \arrow[r, "\top s"] \arrow[d, "r"'] \arrow[dr, "r'"'] & \top M = \mathbb{T}(r') \arrow[d, "\pi"] \arrow[dr, "u"] & \\
P \arrow[r, "i_1"'] \arrow[rr, bend right, "\mathrm{id}_P"'] & P \oplus (Y, y) \arrow[r, "p_1"'] & P
\end{tikzcd}$$

□

Corollary 2.2.15. *There is a natural zigzag homotopy equivalence*

$$\begin{tikzcd} \mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "V^0\Omega K"{name=U}] \arrow[r, bend right, "K\Omega"'{name=D}] & \mathbf{Spectra} \arrow[from=U, to=D, phantom, "\wr\, \simeq"] \end{tikzcd}.$$

Proof. We have the following zigzag of natural homotopy (zigzag) equivalences:

$$V^0\Omega K \xRightarrow[2.2.3]{\simeq} \Omega V^1 K \xLeftarrow[2.2.10]{\simeq} \Omega K q C_x \xLeftarrow[2.2.14]{\simeq} \Omega K t B^b \underset{1.6.5.3}{=\!=\!=} \Omega K q B^b \underset{2.1.6}{\overset{\simeq}{\rightsquigarrow}} K\Omega \qquad \square$$

If we apply π_0 to the zigzag homotopy equivalence $V^0\Omega K\mathcal{N} \sim K\Omega\mathcal{N}$ we get $K_1\mathcal{N} \cong K_0\Omega\mathcal{N}$. So to get a group presentation of $K_1\mathcal{N}$ it is enough to get one for $K_0\Omega\mathcal{N}$. Instead of doing so now, we will first iterate the loop construction and we will obtain a similar isomorphism from which we will extract a group presentation for $K_n\mathcal{N}$ for any n.

Chapter 3

Iteration of the construction

Let $\mathcal{N}$ be an exact category that supports long exact sequences.

3.1 Iterated homotopy cofibers and multi-relative K-theory

For the sake of clarity, we will first discuss the concepts of this section in dimension 2.

Definition 3.1.1. Consider the following commutative square in the category of spectra:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\
X_1' \arrow[r] & X_2'
\end{tikzcd} \tag{3.1}$$

We take horizontal homotopy cofibers and their induced map:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & C \arrow[d, dotted] \\
X_1' \arrow[r] & X_2' \arrow[r] & C'
\end{tikzcd}$$

We define the *iterated homotopy cofiber* of the diagram (3.1) as the homotopy cofiber of the dashed map.

If we are given a morphism of commutative squares of spectra, there is an induced morphism on their iterated homotopy cofibers making the obvious square commute, in such a way that iterated homotopy cofibers define a functor $\mathrm{Arr}^2(\mathbf{Spectra}) \to \mathbf{Spectra}$.

This definition is reasonable by virtue of the following

Remark 3.1.2. We might just as well have taken the homotopy cofibers "vertically" and then taken the homotopy cofiber of the resulting horizontal map. The result is the same up to homotopy equivalence. This is theorem [11, (8.57)] on homotopy equivalence of iterated homotopy cofibers, a result we have already cited. Moreover, this homotopy equivalence is natural, in the sense that if there is a morphism between commutative squares and we take homotopy cofibers vertically and horizontally in both squares, then the two homotopy equivalences make the obvious square commute.

Observe that any functor $F : \mathcal{C} \to \mathcal{D}$ defines an obvious functor $F : \operatorname{Arr}^i(\mathcal{C}) \to \operatorname{Arr}^i(\mathcal{D})$ for any $i \geq 0$.

Definition 3.1.3. We define the *multi-relative K-theory* functor $K : \operatorname{Arr}^2(\mathbf{Wald}) \to \mathbf{Spectra}$ as the composition

$$\operatorname{Arr}^2(\mathbf{Wald}) \xrightarrow{K} \operatorname{Arr}^2(\mathbf{Spectra}) \longrightarrow \mathbf{Spectra}$$

where the second arrow is the iterated homotopy cofiber functor of definition 3.1.1.

More explicitely, given a commutative square in **Wald**:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathcal{C}' \arrow[r] & \mathcal{D}'
\end{tikzcd}$$

its multi-relative K-theory spectrum is the iterated homotopy cofiber of the induced commutative square in K-theory spectra.

The previous definitions and remark can be extended inductively to n-dimensional commutative cubes, giving a multi-relative K-theory functor $K : \operatorname{Arr}^n(\mathbf{Wald}) \to \mathbf{Spectra}$ for any $n \geq 0$. For example, for a three-dimensional commutative cube of spectra, we take homotopy cofibers in one out of the three possible directions. This results in a commutative square of cofibers. We now take the iterated homotopy cofiber as in the previous definition. It can also be proven that it does not depend on the choice of original direction.

Remark 3.1.4. There are multi-relative K-functors $K : \operatorname{Arr}^n(\mathbf{Exact}) \to \mathbf{Spectra}$ for any $n \geq 0$, defined by the inclusion $\mathbf{Exact} \hookrightarrow \mathbf{Wald}$ that considers an exact category as a Waldhausen category with the isomorphisms as quasi-isomorphisms (example 1.2.4)

3.2 Definition of Ω^n

We defined Ω as the functor $\Delta^q : \mathbf{Exact_{les}} \to \operatorname{Arr}(\mathbf{Exact_{les}})$. We now define Ω^n for any $n \geq 2$. For the sake of clarity let us first define it for $n = 2$.

Recall that we have functors $C^q, B^q : \mathbf{Exact_{les}} \to \mathbf{Exact_{les}}$ (remarks 1.5.10.1, 1.6.6.1). In particular, we have $\Omega C^q \mathcal{N} = \Delta : C^q C^q \mathcal{N} \to B^q C^q \mathcal{N}$ and $\Omega B^q \mathcal{N} = \Delta : C^q B^q \mathcal{N} \to B^q B^q \mathcal{N}$.

Taking this into consideration, we make the following

Definition 3.2.1. We define $\Omega^2 \mathcal{N}$ to be the following commutative square in $\mathbf{Exact_{les}}$, i.e., the following object of $\operatorname{Arr}^2(\mathbf{Exact_{les}})$, depicted as an arrow in $\operatorname{Arr}(\mathbf{Exact_{les}})$ on the right:

$$\begin{tikzcd}
C^q C^q \mathcal{N} \arrow[r, "\Delta"] \arrow[d, "C^q\Delta"'] & B^q C^q \mathcal{N} \arrow[d, "B^q\Delta"] & & \Omega C^q \mathcal{N} \arrow[d] \\
C^q B^q \mathcal{N} \arrow[r, "\Delta"'] & B^q B^q \mathcal{N} & = & \Omega B^q \mathcal{N}
\end{tikzcd} \tag{3.2}$$

We have a functor $\Omega^2 : \mathbf{Exact_{les}} \to \operatorname{Arr}^2(\mathbf{Exact_{les}})$, defined obviously on arrows, and thus a functor $K\Omega^2 : \mathbf{Exact_{les}} \to \mathbf{Spectra}$.

More explicitely, we have the following commutative diagram where both horizontal lines are homotopy fibration sequences:

$$\begin{tikzcd}
KC^qC^q\mathcal{N} \arrow[r, "K\Delta"] \arrow[d, "KC^q\Delta"'] & KB^qC^q\mathcal{N} \arrow[r] \arrow[d, "KB^q\Delta"] & K\Omega C^q\mathcal{N} \arrow[d] \\
KC^qB^q\mathcal{N} \arrow[r, "K\Delta"'] & KB^qB^q\mathcal{N} \arrow[r] & K\Omega B^q\mathcal{N}
\end{tikzcd}$$

The homotopy cofiber of the right vertical map is by definition $K\Omega^2\mathcal{N}$. We thus have a homotopy fibration sequence:

$$K\Omega C^q\mathcal{N} \longrightarrow K\Omega B^q\mathcal{N} \longrightarrow K\Omega^2\mathcal{N} \tag{3.3}$$

Remark 3.2.2. Let us describe more explicitely the terms of the homotopy fibration sequence (3.2). For example, take $C^qB^q\mathcal{N}$. Its objects are pairs (N, d) where N is a graded object of $B^q\mathcal{N}$ and d is an acyclic differential on N.

Explicitely, such an object is a $\mathbb{Z}^2$-graded object of $\mathcal{N}$, denoted by $(N_i^j)_{i,j\in\mathbb{Z}}$, endowed with acyclic differentials $d_i^j : N_i^j \to N_{i-1}^j$, $\delta_i^j, \delta_i^{j'} : N_i^j \to N_i^{j-1}$ making the following square commute for every $i, j \in \mathbb{Z}$:

$$\begin{tikzcd}
N_i^j \arrow[r, "d_i^j"] \arrow[d, shift right, "\delta_i^{j'}"'] \arrow[d, shift left, "\delta_i^j"] & N_{i-1}^j \arrow[d, shift right, "\delta_{i-1}^{j'}"'] \arrow[d, shift left, "\delta_{i-1}^j"] \\
N_i^{j-1} \arrow[r, "d_i^{j-1}"'] & N_{i-1}^{j-1}
\end{tikzcd}$$

The description of the objects of $C^qC^q\mathcal{N}$, $B^qC^q\mathcal{N}$, $B^qB^q\mathcal{N}$ is analogous.

The definition and description of $\Omega^n : \mathbf{Exact_{les}} \to \mathrm{Arr}^n(\mathbf{Exact_{les}})$ for higher n is analogous: $\Omega^n\mathcal{N}$ is a certain commutative n-dimensional cube of categories that support long exact sequences. Its multi-relative K-theory spectrum $K\Omega^n\mathcal{N}$ is its iterated homotopy cofiber, and it fits into a homotopy fibration sequence as follows:

$$K\Omega^{n-1}C^q\mathcal{N} \longrightarrow K\Omega^{n-1}B^q\mathcal{N} \longrightarrow K\Omega^n\mathcal{N} \tag{3.4}$$

We introduce some terminology:

Definition 3.2.3. Define the exact category $(B^q)^n\mathcal{N}$ for $n \geq 1$ inductively. Its objects are called *acyclic binary multicomplexes of dimension* n in $\mathcal{N}$.

Explicitely, an acyclic binary multicomplex of dimension n is a $\mathbb{Z}^n$-graded object of $\mathcal{N}$ with two parallel acyclic differentials in each of the n possible directions, making every square commute.

A short sequence of acyclic binary multicomplexes of dimension n is exact if and only if it is short exact exact in every direction.

Observe that $(B^q)^n\mathcal{N}$ is the exact category sitting on the terminal vertex of the cube defining $\Omega^n\mathcal{N}$, just as it is the case for $(B^q)^2\mathcal{N}$: see diagram (3.2).

3.3 Group presentation for $K_n\mathcal{N}$

Lemma 3.3.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a homotopy fibration sequence of connective spectra. Suppose that f admits a splitting: there exists a map $r : Y \to X$ such that $rf = \mathrm{id}_X$. Then if $i \geq 0$, the induced sequence*

$$V^0\Omega^i X \xrightarrow{V^0\Omega^i f} V^0\Omega^i Y \xrightarrow{V^0\Omega^i g} V^0\Omega^i Z \tag{3.5}$$

is also a homotopy fibration sequence.

Proof. The functor Ω is an isomorphism in the homotopy category of spectra, hence it preserves homotopy fibration sequences. The functor V^0 respects homotopies, therefore we have a nullhomotopy of the composition in (3.5). Let F be the homotopy fiber of $V^0\Omega^i g$. We wish to see that the induced map $k : V^0\Omega^i X \to F$ is a weak equivalence. Since the spectra at hand are connective, we need only check that k induces an isomorphism on homotopy from level 0 onwards. Let $j \geq 0$.

The existence of a splitting implies that f is injective in homotopy. This forces the connecting homomorphisms of the long exact homotopy sequence of $X \xrightarrow{f} Y \xrightarrow{g} Z$ to be trivial. Thus this long exact sequence is reduced to the following short exact ones for every $m \geq 0$:

$$0 \longrightarrow \pi_m X \xrightarrow{f_*} \pi_m Y \xrightarrow{g_*} \pi_m Z \longrightarrow 0 \tag{3.6}$$

The natural map $V^0\Omega^i X \to \Omega^i X$ induces an isomorphism on homotopy from level 0 onwards. We also have a natural isomorphism $\pi_j(\Omega^i X) \to \pi_{j+i}(X)$. Of course, this also holds for Y and Z. We therefore have a commutative diagram as follows:

$$\begin{tikzcd}
 & \pi_j(V^0\Omega^i X) \arrow[r, "(V^0\Omega^i f)_*"] \arrow[d, "\cong"'] & \pi_j(V^0\Omega^i Y) \arrow[r, "(V^0\Omega^i g)_*"] \arrow[d, "\cong"'] & \pi_j(V^0\Omega^i Z) \arrow[d, "\cong"'] & \\
0 \arrow[r] & \pi_{j+i} X \arrow[r, "f_*"] & \pi_{j+i} Y \arrow[r, "g_*"] & \pi_{j+i} Z \arrow[r] & 0
\end{tikzcd} \tag{3.7}$$

The sequence on the bottom is exact as shown in (3.6). The sequence on the top is therefore a short exact sequence, being isomorphic to the bottom one.

We just proved that $(V^0\Omega^i g)_*$ is surjective for $j \geq 0$, therefore the long exact homotopy sequence for the homotopy fibration $F \to V^0\Omega^i Y \to V^0\Omega^i Z$ also decomposes as short exact sequences. We have then the following commutative diagram displaying two short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \pi_j(V^0\Omega^i X) \arrow[r, "(V^0\Omega^i f)_*"] \arrow[d, "k_*"'] & \pi_j(V^0\Omega^i Y) \arrow[r, "(V^0\Omega^i g)_*"] & \pi_j(V^0\Omega^i Z) \arrow[r] & 0 \\
0 \arrow[r] & \pi_j F \arrow[ur] & & &
\end{tikzcd}$$

Therefore k_* is an isomorphism for every $j \geq 0$, proving the theorem. □

Lemma 3.3.2. *There is a natural homotopy equivalence*

$$\begin{tikzcd}
\mathbf{Spectra} \arrow[r, bend left, "V^0\Omega^i V^0"] \arrow[r, bend right, "V^0\Omega^i"'] \arrow[r, phantom, "\Downarrow\simeq"] & \mathbf{Spectra}
\end{tikzcd}$$

for any $i \geq 0$.

Proof. Consider the natural transformation $p : V^0 \Rightarrow \mathrm{id}_{\mathbf{Spectra}}$ from the definition of the Whitehead tower. We compose it with the functor $V^0\Omega^i$ to get a natural transformation $V^0\Omega^i V^0 \Rightarrow V^0\Omega^i$. It is a homotopy equivalence for every spectrum X.

Indeed, since the map $p_X : V^0X \to X$ induces an isomorphism on homotopy from level 0 onwards, the shifted map $\Omega^i p_X : \Omega^i V^0 X \to \Omega^i X$ induces an isomorphism from level $-i$ onwards. Therefore the map $V^0\Omega^i p_X : V^0\Omega^i V^0 X \to V^0\Omega^i X$ induces an isomorphism from level 0 onwards. Since the V^0 functor yields connective spectra, Whitehead's theorem applies to show that $V^0\Omega^i p_X$ is a homotopy equivalence. ☐

Theorem 3.3.3. *There is a natural zigzag homotopy equivalence*

$$\begin{tikzcd}
\mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "K\Omega^n"] \arrow[r, bend right, "V^0\Omega^n K"'] \arrow[r, phantom, "\wr\,\simeq"] & \mathbf{Spectra}
\end{tikzcd}$$

for any $n \geq 1$.

Proof. We will prove the result by induction. For $n = 1$ this is corollary 2.2.15.

Now suppose we have the result for $n-1$. The naturality assumption applied to the arrow $\Delta : C^q\mathcal{N} \to B^q\mathcal{N}$ in $\mathbf{Exact}_{\mathbf{les}}$ gives a homotopy commutative square (i.e., homotopy commutative at every stage of the zigzag):

$$\begin{tikzcd}
K\Omega^{n-1}C^q\mathcal{N} \arrow[r, "K\Omega^{n-1}\Delta"] \arrow[d, squiggly, no head, "\simeq"'] & K\Omega^{n-1}B^q\mathcal{N} \arrow[d, squiggly, no head, "\simeq"] \\
V^0\Omega^{n-1}KC^q\mathcal{N} \arrow[r, "V^0\Omega^{n-1}K\Delta"'] & V^0\Omega^{n-1}KB^q\mathcal{N}
\end{tikzcd}$$

Both horizontal maps fit into homotopy fibration sequences, as shown in the following homotopy commutative diagram:

$$\begin{tikzcd}
K\Omega^{n-1}C^q\mathcal{N} \arrow[r, "K\Omega^{n-1}\Delta"] \arrow[d, squiggly, no head, "\simeq"'] & K\Omega^{n-1}B^q\mathcal{N} \arrow[r] \arrow[d, squiggly, no head, "\simeq"] & K\Omega^n\mathcal{N} \arrow[d, squiggly, no head, "\simeq"] \\
V^0\Omega^{n-1}KC^q\mathcal{N} \arrow[r, "V^0\Omega^{n-1}K\Delta"'] & V^0\Omega^{n-1}KB^q\mathcal{N} \arrow[r] & V^0\Omega^{n-1}K\Omega\mathcal{N}
\end{tikzcd}$$

The top one is the homotopy fibration sequence (3.4) characterizing multi-relative K-theory. The lower one is obtained by applying $V^0\Omega^{n-1}$ to the top one, as per lemma 3.3.1. The zigzag on the right is obtained by taking the induced map on cofibers at every stage of the zigzag. It is therefore composed of homotopy equivalences. It is natural, too, since Δ and homotopy cofibers are.

We have the following sequence of natural (zigzag) homotopy equivalences, yielding the result:

$$\begin{tikzcd}
K\Omega^n \arrow[r, squiggly, no head, "\simeq"] & V^0\Omega^{n-1}K\Omega \arrow[r, squiggly, no head, "\simeq", "2.2.15"'] & V^0\Omega^{n-1}V^0\Omega K \arrow[r, Rightarrow, "\simeq", "3.3.2"'] & V^0\Omega^n K
\end{tikzcd}$$

☐

Corollary 3.3.4. *There is a natural isomorphism*

$$\begin{tikzcd}
\mathbf{Exact}_{\mathbf{les}} \arrow[r, bend left, "\pi_0 K\Omega^n"] \arrow[r, bend right, "K_n"'] \arrow[r, phantom, "\Downarrow\cong"] & \mathbf{Ab}
\end{tikzcd}$$

for any $n \geq 1$.

Proof. Let $n \geq 1$. We have natural isomorphisms $\pi_0 V^0 \Omega^n \overset{\simeq}{\Longrightarrow} \pi_0 \Omega^n \overset{\simeq}{\Longrightarrow} \pi_n$ of functors **Spectra** $\to$ **Ab**. Precomposing them with K we get a natural isomorphism

$$\begin{tikzcd} \mathbf{Exact_{les}} \arrow[r, bend left, "\pi_0 V^0 \Omega^n K"{name=U}] \arrow[r, bend right, "K_n"'{name=D}] & \mathbf{Ab} \arrow[Rightarrow, from=U, to=D, phantom, "\Downarrow\cong"] \end{tikzcd},$$

recalling that by definition $K_n = \pi_n K$.

We also have a natural isomorphism

$$\begin{tikzcd} \mathbf{Exact_{les}} \arrow[r, bend left, "\pi_0 K \Omega^n"{name=U}] \arrow[r, bend right, "\pi_0 V^0 \Omega^n K"'{name=D}] & \mathbf{Ab} \arrow[Rightarrow, from=U, to=D, phantom, "\Downarrow\cong"] \end{tikzcd}$$

coming from theorem 3.3.3 and remark 1.7.3.

Composing these two yields the desired natural isomorphism. ☐

Our final step is to give a presentation for the functor $\pi_0 K \Omega^n$.

Lemma 3.3.5. *Consider the following commutative diagram of abelian groups with rows and column exact:*

$$\begin{tikzcd} & A \arrow[r, "p"] \arrow[d, "g"'] & B \arrow[r] \arrow[d, "h"] & 0 \\ C \arrow[r, "f"'] & D \arrow[r, "k"'] & E \arrow[r] \arrow[d, "q"] & 0 \\ & & F \arrow[d] & \\ & & 0 & \end{tikzcd}$$

Then the map $qk : D \to F$ descends to an isomorphism $D/H \to F$, where H is the subgroup of D generated by the images of f and g.

This isomorphism is natural, in the sense that if we have another such diagram and a morphism of diagrams between them then the two isomorphisms make the induced square commute.

Proof. Let us prove that the kernel of qk is H. First observe that, by exactness, $qkf = 0$ and $qkg = qhp = 0$. Thus H is included in the kernel of qk. To prove the reverse inclusion, we consider $d \in D$ such that $qk(d) = 0$. Then $k(d) = h(b)$ for some $b \in B$. Also $b = p(a)$ for some $a \in A$. Thus $k(d) = hp(a) = kg(a)$. Hence there exists a $c \in C$ such that $d - g(a) = f(c)$, and thus $d = g(a) + f(c) \in H$, proving the statement.

Since qk is an epimorphism, this proves that the map $qk : D \to F$ passes to the quotient as an isomorphism $\varphi : D/H \to F$.

Let us now check naturality, which is the commutativity of the bottom square in the following diagram:

$$\begin{tikzcd}
 & & & A' \arrow[rr] \arrow[dl] \arrow[dd, "g'"] & & B' \arrow[dl] \arrow[dd] & & \\
 & & A \arrow[rr] \arrow[dd, "g"] & & B \arrow[dd] & & & \\
 & C' \arrow[rr, "f'"] \arrow[dl] & & D' \arrow[rr, "k'"] \arrow[dl, "d"] \arrow[dd, "\pi'"] & & E' \arrow[rr] \arrow[dl, "e"] \arrow[dd, "q'"] & & 0 \\
C \arrow[rr, "f"'] & & D \arrow[rr, "k"] \arrow[dd, "\pi"] & & E \arrow[rr] \arrow[dd, "q"] & & 0 & \\
 & & & D'/H' \arrow[rr, "\varphi'"] \arrow[dl, dotted, "\tilde{d}"] & & F' \arrow[dl, "r"] \arrow[dd] & & \\
 & & D/H \arrow[rr, "\varphi"'] & & F \arrow[dd] & & & \\
 & & & & & 0 & & \\
 & & & & 0 & & &
\end{tikzcd}$$

First, observe that the map d descends to a map $\tilde{d}$, since $dg'(A') \subset g(A)$ and $df'(C') \subset f(C)$ by commutativity. Since π' is a quotient map, it is surjective and thus it suffices to see that $\varphi\tilde{d}\pi' = r\varphi'\pi'$. We check this:

$$\varphi\tilde{d}\pi' = \varphi\pi d = qkd = qek' = rq'k' = r\varphi'\pi' \qquad \square$$

Corollary 3.3.6. *Consider the following commutative square of connective spectra:*

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[d, "g"] \\
X_3 \arrow[r, "f"'] & X_4
\end{tikzcd}$$

Take its iterated homotopy cofiber, denoted by $\bar{X}$:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d, "g"] & A \arrow[d] \\
X_3 \arrow[r, "f"'] & X_4 \arrow[r, "k"'] & B \arrow[d, "q"] \\
 & & \bar{X}
\end{tikzcd}$$

Take long exact homotopy sequences:

$$\begin{tikzcd}
\pi_0 X_1 \arrow[r] \arrow[d] & \pi_0 X_2 \arrow[r] \arrow[d, "g_*"] & \pi_0 A \arrow[r] \arrow[d] & 0 \\
\pi_0 X_3 \arrow[r, "f_*"'] & \pi_0 X_4 \arrow[r, "k_*"'] & \pi_0 B \arrow[r] \arrow[d, "q_*"] & 0 \\
 & & \pi_0 \bar{X} \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

Then the map $q_* k_* : \pi_0 X_4 \to \pi_0 \bar{X}$ *descends to an isomorphism* $\pi_0 X_4/H$ *where* H *is the subgroup of* $\pi_0 X_4$ *generated by the images of* f_* *and* g_*.

This isomorphism is natural, in the sense that if we have a commutative cube of connective spectra

$$\begin{tikzcd}
 & X_1' \arrow[rr] \arrow[dl] \arrow[dd] & & X_2' \arrow[dl] \arrow[dd, "g'"] \\
X_1 \arrow[rr] \arrow[dd] & & X_2 \arrow[dd, "g"] & \\
 & X_3' \arrow[rr, "f'"] \arrow[dl] & & X_4' \arrow[dl, "d"] \\
X_3 \arrow[rr, "f"'] & & X_4 &
\end{tikzcd}$$

then the induced map $\widetilde{d_*} : (\pi_0 X_4)/H \to (\pi_0 X_4')/H'$ *and the induced map on the* π_0 *of iterated homotopy cofibers* $\pi_0 \bar{X}' \to \pi_0 \bar{X}$ *make the following square commute:*

$$\begin{tikzcd}
(\pi_0 X_4')/H' \arrow[r, "\cong"] \arrow[d, "\widetilde{d_*}"'] & \pi_0 \bar{X}' \arrow[d] \\
(\pi_0 X_4)/H \arrow[r, "\cong"'] & \pi_0 \bar{X}
\end{tikzcd}$$

Corollary 3.3.7. *The abelian group* $K_n\mathcal{N}$ *admits the following presentation. It has one generator per acyclic binary multicomplex of dimension* n *in* $\mathcal{N}$*, and these are subject to the following relations:*

1. $[N] = [N'] + [N'']$ *if there is a short exact sequence* $0 \to N' \to N \to N'' \to 0$ *in* $(B^q)^n\mathcal{N}$,
2. *any acyclic binary multicomplex of dimension* n *that has some pair of parallel differentials equal is made to vanish.*

Moreover, if $F : \mathcal{M} \to \mathcal{N}$ *is an arrow in* $\mathbf{Exact}_{\mathbf{les}}$*, then with the above presentation the abelian group homomorphism* $K_nF : K_n\mathcal{M} \to K_n\mathcal{N}$ *acts as* $K_nF([M]) = [(B^q)^n F(M)]$ *for any* M *acyclic binary multicomplex of dimension* n *in* $\mathcal{M}$.

Proof. For the sake of clarity we will prove it in the case where $n = 2$. The general case is proven analogously after suitable generalizations of lemma 3.3.5 and corollary 3.3.6.

Recall that the multi-relative K-theory spectrum $K\Omega^2\mathcal{N}$ is the iterated homotopy cofiber of the following commutative square:

$$\begin{tikzcd}
KC^qC^q\mathcal{N} \arrow[r, "K\Delta"] \arrow[d, "KC^q\Delta"'] & KB^qC^q\mathcal{N} \arrow[d, "KB^q\Delta"] \\
KC^qB^q\mathcal{N} \arrow[r, "K\Delta"'] & K(B^q)^2\mathcal{N}
\end{tikzcd} \tag{3.8}$$

Let $F : \mathcal{M} \to \mathcal{N}$ be an arrow in $\mathbf{Exact_{les}}$. Combining corollary 3.3.6 with corollary 3.3.4 and the fact that $\pi_0 K \cong K_0$, we get the following commutative diagram, where the notation $A/\langle f, g\rangle$ means the quotient group of A by the subgroup generated by the images of f and g:

$$\begin{tikzcd}
K_0(B^q)^2\mathcal{M} \arrow[r, "\pi'"] \arrow[d, "K_0(B^q)^2F"'] & \frac{K_0(B^q)^2\mathcal{M}}{\langle K_0B^q\Delta, K_0\Delta\rangle} \arrow[r, "\cong"] \arrow[d, "K_0\widetilde{(B^q)}^2F"'] & \frac{\pi_0K(B^q)^2\mathcal{M}}{\langle (KB^q\Delta)_*, (K\Delta)_*\rangle} \arrow[r, "\cong"] \arrow[d, "(K(B^q)^2F)_*"'] & \pi_0K\Omega^2\mathcal{M} \arrow[r, "\cong"] \arrow[d, "(K\Omega^2F)_*"'] & K_2\mathcal{M} \arrow[d, "K_2F"] \\
K_0(B^q)^2\mathcal{N} \arrow[r, "\pi"'] & \frac{K_0(B^q)^2\mathcal{N}}{\langle K_0B^q\Delta, K_0\Delta\rangle} \arrow[r, "\cong"'] & \frac{\pi_0K(B^q)^2\mathcal{N}}{\langle (KB^q\Delta)_*, (K\Delta)_*\rangle} \arrow[r, "\cong"'] & \pi_0K\Omega^2\mathcal{N} \arrow[r, "\cong"'] & K_2\mathcal{N}
\end{tikzcd}$$

In it we read off an isomorphism $\frac{K_0(B^q)^2\mathcal{N}}{\langle K_0B^q\Delta, K_0\Delta\rangle} \to K_2\mathcal{N}$. Now recall proposition 1.7.4: it gives an explicit presentation for the K_0 groups and an explicit description of the K_0 homomorphisms. This yields the desired presentation of $K_2\mathcal{N}$, since indeed the images of $K_0B^q\Delta$ and $K_0\Delta$ give the acyclic binary multicomplexes of dimension 2 that have some pair of parallel differentials equal.

From the same proposition and the above commutative ladder we also get the desired description of K_2F. ☐

Corollary 3.3.7 remains valid if $\mathcal{N}$ does not support long exact sequences. We refer the reader to section 6 of [4] for details. Grayson remarks that in this generality, quasi-isomorphisms in $C\mathcal{N}$ might not be closed under composition, but still $C^q\mathcal{N}$, the full subcategory of $C\mathcal{N}$ consisting of acyclic complexes, is an exact category if $\mathcal{N}$ is. He then goes on to generalize corollary 2.2.15 to arbitrary exact categories. The main ingredient is the cofinality theorem combined with the fact that there is a natural way to embed an exact category as a full subcategory, closed under extensions, of an exact category that supports long exact sequences.

We point out two follow-ups to the work we exposed in this dissertation.

Grayson [5] generalized the presentation for the higher K-groups of an exact category in corollary 3.3.7 to the relative K-groups of an exact functor between exact categories. From this he derived the long exact sequence of K-theory groups, without using homotopy theory.

Harris [6] gave proofs of the additivity, resolution and cofinality theorems in K-theory by adopting Grayson's presentation as a definition, without using any homotopy theory either.

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