LURIE'S CONSTRUCTION OF THE SMASH PRODUCT OF SPECTRA

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ABSTRACT. These are notes for a talk in which I present Lurie's construction of the smash product of spectra and some related results to an audience with a casual acquaintance with ∞-category theory.

CONTENTS

1. HISTORICAL INTRODUCTION

In order to appreciate why a construction of a good smash product of spectra is a big deal, it's useful to recall some history.

By the late 1960s, there was an established answer to what is the right object that deserves to be called the "stable homotopy category", or "the homotopy category of spectra" $\mathcal{SHC}$. Michael Boardman was the one to give the first construction on some mimeographed notes, see also [Vog70]. Spanier and Whitehead had constructed their eponymous category in 1953, but it was deficient in that not every cohomology theory on spaces could be represented in it, for example.

What was now needed was a *smash product* in order to reflect the multiplicativity of some cohomology theories at the level of their representing spectra. It's not immediately obvious how to build such a smash product in $\mathcal{SHC}$: if X and Y are spectra, then how to make a spectrum out of $\{X_n \wedge Y_m\}_{n,m\geq 0}$? The reader inspired by chain complexes could think that a similar solution would work here. But there seems to be no way of defining the structure maps that doesn't depend of making some arbitrary choices and, moreover, the unit would be forced to be the trivial spectrum instead of the sphere spectrum $\mathbb{S}$, which is not a behavior we want...

The solution found by Adams [Ada74] was to consider any cofinal choice of a poset $\mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$: these are his famous "handicrafted smash products". They furnish $\mathcal{SHC}$ with a symmetric monoidal structure with unit $\mathbb{S}$, all right. But before taking homotopy, that is, in the category of spectra, it's not associative.

It's useful to use the vocabulary from abstract homotopy theory to phrase what it is exactly that we want. I might as well just quote May [May99]:

> The ideal category of spectra should be a complete and cocomplete Quillen model category, tensored and cotensored over the category of based spaces (or simplicial sets), and closed symmetric monoidal under the smash product. Its

homotopy category (obtained by inverting the weak equivalences) should be equivalent to Mike [Boardman]'s original stable homotopy category.

If we had such an ideal category of spectra, then ideally we'd also get a model category of A_∞ or E_∞-ring spectra, allowing us to do homotopy theory with them (e.g. consider homotopy (co)limits of those, which is not possible at the level of the homotopy category).

It took some thirty years to get such ideal categories: see [EKMM97], [HSS00], [MMSS01]. EKMM's S-modules make use of some clever observation of Hopkins about the linear isometries operad. It builds on the category of coordinate-free May spectra from [LMSM86]. The categories of symmetric spectra and orthogonal spectra look more similar to the classical (pre)spectra, but their component spaces are endowed with actions by the symmetric or orthogonal groups. The categorical construction given by the Day convolution product is used to get the smash product.

As attested by this long quest for a good smash product, there's nothing obvious about its construction or even its existence. That's why a radically different approach to it, some 15 years after the first construction and with some new technology, is worthy of notice.

2. A layout of Lurie's approach

Before we start the exposition proper, let's make a summary of the approach.

In my opinion, the biggest conceptual difference in this construction is that it does not proceed by looking at some category of spectra and building a monoidal product for it by hand. Instead, we will consider the ∞-category of spectra Sp as an object inside a bigger ∞-category of ∞-categories. The most fundamental property of Sp is that it is *stable*, and we will use that Sp is indeed very special among stable ∞-categories. The emergence of Sp as a symmetric monoidal ∞-category follows from the fact that stable ∞-categories are a smashing localization of ∞-categories.

Ultimately, it boils down to this: monoidal units $\mathbf{1}$ of symmetric monoidal ∞-categories are canonically commutative algebras therein, in a very simple way: the multiplication is the natural equivalence $\mathbf{1} \otimes \mathbf{1} \simeq \mathbf{1}$ and the unit is the identity. So it's enough to find some symmetric monoidal ∞-category whose commutative algebras are exactly closed symmetric monoidal ∞-categories, and which has Sp as its monoidal unit.

This is all quite vague: let's make it a little less vague.

(1) First, a technical point: we don't consider all ∞-categories but only the presentable ones together with colimit-preserving functors. These form the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.
(2) Then, we will observe that the subcategory $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}$ of stable presentable ∞-categories is a reflective localization of it. The localization functor is given by $\mathrm{Sp}(-)$, the functor taking spectrum objects.
(3) We will now see the ∞-category Sp being singled out: the functor $\mathrm{Sp}(-)$ is equivalent to $\mathrm{Sp} \otimes -$. This $\otimes$ is a symmetric monoidal product on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ such that $\mathcal{C} \otimes \mathcal{C}'$ corepresents the functor sending $\mathcal{D}$ to the ∞-category $\mathrm{Fun}^{\mathrm{L,L}}(\mathcal{C} \times \mathcal{C}', \mathcal{D})$ of functors preserving colimits separately in each variable.
(4) Therefore, the localization functor $\mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ has a special form: $\mathrm{Sp} \otimes -$. This is equivalent, by some general theory, to Sp being an idempotent object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ under $\otimes$: essentially, $\mathrm{Sp} \otimes \mathrm{Sp} \simeq \mathrm{Sp}$.
(5) This idempotency property endows Sp with the structure of a commutative algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Unfolding the definitions, a commutative algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is precisely a closed symmetric monoidal presentable ∞-category. This gets us the result.
(6) A universal property is established along the way, namely: $(\mathrm{Sp}, \wedge, \mathbb{S})$ is initial among stable presentable closed symmetric monoidal ∞-categories. This follows from the

equivalence of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ with $\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, i.e. the ∞-category of modules over $(\mathrm{Sp}, \wedge, \mathbb{S}) \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

(7) In conclusion, the smash product in Sp is completely determined by asking that it determine a closed symmetric monoidal structure, and that $\mathbb{S}$ be the unit.

3. Presentable ∞-categories

Instead of working with model categories as a device for encoding a homotopy theory, Lurie [Lur09b], [Lur17] uses ∞-categories, also known as quasicategories or weak Kan complexes. The ∞-category of spaces[1] is denoted by $\mathcal{S}$.

We shall be interested in a special class of ∞-categories called *presentable*. The 1-categorical analog is more often called "locally presentable". There are several ways to understand this important concept; a short introduction can be found in [Gro20]. One way of formulating a definition is: $\mathcal{C}$ is presentable if it is cocomplete, locally small [Lur09b, 5.4.1.7] and there exists a regular cardinal κ and a set S of κ-compact objects of $\mathcal{C}$ such that every object of $\mathcal{C}$ is a colimit of a small diagram taking values in the full subcategory of $\mathcal{C}$ spanned by S [Lur09b, 5.5.1.1]. Here, a *κ-compact object* is an object $c \in \mathcal{C}$ such that $\mathcal{C}(c, -) : \mathcal{C} \to \mathcal{S}$ preserves κ-filtered colimits. It's easiest to think of the case $\kappa = \aleph_0$, in which case the 1-categorical analogue is called a *locally finitely presentable category*.

The ∞-category of spaces $\mathcal{S}$ is presentable: we can take $S = \{*\}$, since for $X \in \mathcal{S}$, we have $X \simeq \mathrm{colim}(X \xrightarrow{\{*\}} \mathcal{S})$. This is an ∞-categorical analog of the fact that every set is a colimit of the set with one element, which is a particular case of the fact that every presheaf is a colimit of representable ones.

A couple of important facts: presentable ∞-categories are complete [Lur09b, 5.5.2.4]. A version of the adjoint functor theorem says that if $F : \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories and $\mathcal{C}$ is presentable, then F is a left adjoint as soon as it preserves colimits [Lur09b, 5.5.2.9/10].

Of special interest to us will be the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ with objects given by (small)[2] presentable ∞-categories and arrows given by colimit-preserving functors. I've informally heard, though, that presentability is not actually needed for what's presented in this note, and that the main results go through if you consider the larger ∞-category of all cocomplete ∞-categories with colimit-preserving functors.

4. Localizations

The word "localization" is often used in the following sense: you have a category $\mathcal{C}$ and a class of arrows S in it, and you want to construct a category $S^{-1}\mathcal{C}$ which is the universal category with the arrows in S inverted. Sometimes, the category $S^{-1}\mathcal{C}$ is actually a subcategory of $\mathcal{C}$, in which case the localization is called "reflective". For example, if $f : R \to S$ is an epimorphism of commutative rings (e.g. a quotient or a localization), then the extension of scalars functor $S \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_S$ is a reflective localization [sta, 08YS]: Mod_S can be identified with a subcategory of Mod_R. For example, if $f : \mathbb{Z} \to \mathbb{Q}$, then this subcategory consists of the torsion-free divisible abelian groups. The functor $- \otimes \mathbb{Q} : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Q}}$ is the localization at the morphisms of abelian groups $g : A \to B$ such that $g \otimes \mathbb{Q}$ is an isomorphism.

Lurie reserves the word "localization" for a reflective localization, so we shall do the same. In other words:

Definition 4.1. A functor of ∞-categories $L : \mathcal{C} \to \mathcal{D}$ is a *localization* if it has a fully faithful right adjoint i[3]. We will denote by $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow iL$ a unit for this adjunction.

[1] Or "anima" in the terminology of Peter Scholze.

[2] We shall generally ignore size considerations.

[3] Which we will often omit from the notation, as we may think of i as the inclusion of a subcategory.

See [Lur09b, 5.2.7.4] and [Lur, 02F5] for alternative formulations and properties.

4.1. **Smashing localizations.** There is a special type of localization of symmetric monoidal ∞-categories which has particularly good properties.

Definition 4.2. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal ∞-category. An object $E \in \mathcal{C}$ is *idempotent* if there exists a morphism $e : \mathbf{1} \to E$ inducing an equivalence

$$E \simeq \mathbf{1} \otimes E \xrightarrow{e \otimes \mathrm{id}} E \otimes E.^{4}$$

Proposition 4.3. [Lur17, 4.8.2.4/7] *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal ∞-category and $E \in \mathcal{C}$. Then*

$$L := E \otimes - : \mathcal{C} \to \mathcal{C}$$

is a localization onto its essential image $L\mathcal{C}$ if and only if E is idempotent. The object E is idempotent via $e : \mathbf{1} \to E$ if and only if the unit of the adjunction (L, i) is of the form $e \otimes \mathrm{id} : \mathrm{id}_{\mathcal{C}} \Rightarrow iL$.

In this situation, $L\mathcal{C}$ can be endowed with a symmetric monoidal structure $(L\mathcal{C}, \otimes, L\mathbf{1} \simeq E)$ such that $L : \mathcal{C} \to L\mathcal{C}$ is symmetric monoidal, its right adjoint i is lax symmetric monoidal, and it satisfies that $i(d) \otimes i(d') \simeq i(d \otimes d')$.[5]

Remark 4.4. Knowing that $(L = E \otimes -, i)$ is a smashing localization with unit η we can recover $e : \mathbf{1} \to E$ as $\eta_{\mathbf{1}} : \mathbf{1} \to E \otimes \mathbf{1} \simeq E$.

A localization as above is called a *smashing localization*. This is the terminology of [Rav84].

The above properties imply that there is an induced adjunction $\mathrm{CAlg}(\mathcal{C}) \underset{i}{\overset{L}{\rightleftarrows}} \mathrm{CAlg}(L\mathcal{C})$ with i fully faithful. Now, $E \simeq L\mathbf{1}$ is the monoidal unit of $L\mathcal{C}$, therefore it is a commutative algebra in $L\mathcal{C}$ in a canonical way, and therefore (via i) it is a commutative algebra in $\mathcal{C}$ with unit $e : \mathbf{1} \to E$.

Since

$$E \simeq \mathbf{1} \otimes E \xrightarrow{e \otimes \mathrm{id}} E \otimes E \xrightarrow{\mu} E^{6} \tag{4.5}$$

is equivalent to the identity, the 2-out-of-3 property of weak equivalences gives that E being a commutative algebra which is idempotent via its unit is equivalent to E being an idempotent commutative algebra in $\mathcal{C}$:

Definition 4.6. An *idempotent* (or *solid*[7]) commutative algebra in a symmetric monoidal ∞-category $\mathcal{C}$ is an $E \in \mathrm{CAlg}(\mathcal{C})$ such that its multiplication map is an equivalence.

Proposition 4.7. *The functor $\mathrm{CAlg}^{\mathrm{idem}}(\mathcal{C}) \to \mathcal{C}_{\mathbf{1}/}$ taking an idempotent commutative algebra E to its unit $\mathbf{1} \to E$ is an equivalence onto the subcategory of idempotent objects of $\mathcal{C}$.*

Proof. We have just explained why it's essentially surjective – for the full faithfulness, see [Lur17, 4.8.2.9]. $\square$

In particular: if we are given an object $c \in \mathcal{C}$ with a morphism $\mathbf{1} \to c$ which exhibits it as an idempotent object, then there exists an essentially unique (idempotent) commutative algebra structure in c which has $\mathbf{1} \to c$ as its unit.

[4] This is sloppy: an idempotent object should be the data of such a morphism [Lur17, 4.8.2.1/6]. But allow me to be sloppy for the sake of conciseness. Also, note that this is equivalent to $\mathrm{id} \otimes e$ being an equivalence, as follows from (4.5) and the other unit axiom.

[5] This is not saying that i is symmetric monoidal as it typically doesn't preserve the monoidal unit.

[6] Here μ denotes the multiplication of $E \in \mathrm{CAlg}(\mathcal{C})$, which is simply $L\mathbf{1} \otimes L\mathbf{1} \simeq L\mathbf{1}$, since $L\mathbf{1}$ is the monoidal unit in $\mathrm{CAlg}(L\mathcal{C})$ and i preserves the monoidal product.

[7] This is the terminology of [BK72].

Here's one partial, but concise corollary of the above, which hides all idempotency considerations:

Corollary 4.8. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $E \in \mathcal{C}$. If $E \otimes - : \mathcal{C} \to \mathcal{C}$ is a localization onto its essential image, then E can be endowed with the structure of a commutative algebra in $\mathcal{C}$.*

If we are a bit more careful, we also get a uniqueness result:

Corollary 4.9. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $E \in \mathcal{C}$. Suppose $L := E \otimes - : \mathcal{C} \to \mathcal{C}$ is a localization onto its essential image, with unit of the (L, i) adjunction given by $e \otimes \mathrm{id} : \mathrm{id}_{\mathcal{C}} \Rightarrow Li$ for some $e : \mathbf{1} \to E$. Then E has an essentially unique structure of a commutative algebra whose unit is e.*

The following is an enlightening reinterpretation of these localizations as extension – restriction of scalars between ∞-categories of modules[8], which gets us a universal property.

Proposition 4.10. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal ∞-category, $E \in \mathrm{CAlg}(\mathcal{C})$ an idempotent commutative algebra, and $L := E \otimes - : \mathcal{C} \to L\mathcal{C}$ be the localization functor from Proposition 4.3. We have the following vertical equivalence of adjunctions of ∞-categories*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "E\otimes -"] \arrow[r, phantom, "\perp"] & L\mathcal{C} \arrow[l, hook, shift left] \\
\mathrm{Mod}_{\mathbf{1}}(\mathcal{C}) \arrow[u, "U"', "\sim"] \arrow[r, shift left, "E\otimes -"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_E(\mathcal{C}) \arrow[u, "\sim"'] \arrow[l, shift left, "\mathrm{res}^E_{\mathbf{1}}"]
\end{tikzcd}$$

where U is the forgetful functor and res *denotes restriction of scalars. These equivalences are moreover symmetric monoidal, where $\mathrm{Mod}_E(\mathcal{C})$ has a symmetric monodial structure with monoidal product $\otimes$ and monoidal unit E. So $\mathrm{CAlg}(L\mathcal{C}) \simeq \mathrm{CAlg}(\mathrm{Mod}_E(\mathcal{C})) = \mathrm{CAlg}_E(\mathcal{C})$.*

In particular, if $B \in \mathrm{CAlg}(L\mathcal{C})$, then there is an essentially unique morphism $E \to B$ of commutative algebras in $\mathcal{C}$. [9]

4.2. **Stable ∞-categories.** Recall that an ∞-category is called *stable* if it is pointed (has a zero object), has finite limits and colimits, and a commutative square is a pushout iff it is a pullback [Lur17, 1.1.3.4].

Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ generated by the stable ∞-categories. Happily, if an ∞-category is stable, then presentability is equivalent to an easier set of conditions [Lur17, 1.4.4.2].

There is a functor

$$\mathrm{Sp}(-) : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$$

of (Ω-)*spectra objects*. One can define this functor in a very similar fashion to classical Ω-spectra, which would be $\mathrm{Sp}(\mathcal{S}) =: \mathrm{Sp}$, see [Lur09a, 8.4]. In Higher Algebra, Lurie dumped this definition in favor of an equivalent, more abstract formulation, more in tune with the Goodwillie calculus philosophy, see [Lur17, 1.4.2.8].[10]

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[8]Note that we need to start with the knowledge that E has a commutative algebra structure, so that E-modules make sense.

[9]This follows from E being initial in $\mathrm{CAlg}_E(\mathcal{C})$, plus the fact that the forgetful functor $\mathrm{CAlg}_E(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$ is fully faithful.

[10]That would be $\mathrm{Exc}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$, the ∞-category of reduced, excisive functors from pointed finite spaces to $\mathcal{C}$. The two approaches are equivalent by [Lur09a, 10.16]; see also 10.18 which suggests to think of a reduced, excisive functor as a (space-valued) homology theory on finite spaces: taking its homotopy groups gets us closer to the classical notion of an (abelian group-valued) homology theory.

Now, $\mathrm{Sp}(\mathcal{C})$ is stable as soon as $\mathcal{C}$ has finite limits [Lur17, 1.4.2.17] and it is the universal stable ∞-category associated to $\mathcal{C}$ [Lur17, 1.4.2.23]. With presentability hypotheses we have a universal property that we shall be more interested in, so let's phrase it. First, if $\mathcal{C}$ is presentable then $\mathrm{Sp}(\mathcal{C})$ is presentable [Lur17, 1.4.4.4], and we have a functor $\mathrm{Sp} : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$. Second, note that we have a functor $\Sigma^\infty_+ : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ which assigns to an object its suspension spectrum. It is the component at $\mathcal{C}$ of the unit of a localization adjunction:

Proposition 4.11. [Lur17, 1.4.4.5] *Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories and suppose that $\mathcal{D}$ is stable. The following functor is an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow[\sim]{(\Sigma^\infty_+)^*} \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$

Thus [Lur, 6.2.4.5] *the functor* $\mathrm{Sp}(-) : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ *is a localization*

$$\begin{tikzcd} \mathcal{P}\mathrm{r}^{\mathrm{L}} \arrow[r, shift left, "\mathrm{Sp}(-)"] \arrow[r, phantom, "\perp"] & \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}} \arrow[l, shift left, hook] \end{tikzcd}$$

with adjunction unit evaluated in $\mathcal{C}$ given by $\Sigma^\infty_+ : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$.

The *sphere spectrum* $\mathbb{S}$ is the image of the one-point space by the functor $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$.

We shan't use the following corollary, but we can't refrain from mentioning it, as it gives a universal property for Sp.

Corollary 4.12. *The ∞-category* Sp *is the stable presentable ∞-category freely generated by a single object. More precisely, if $\mathcal{D}$ is a presentable stable ∞-category, then the evaluation functor on the sphere spectrum*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathcal{D}) \xrightarrow[\sim]{\{\mathbb{S}\}^*} \mathrm{Fun}(*, \mathcal{D}) \simeq \mathcal{D}$$

is an equivalence of ∞-categories. Here $\{\mathbb{S}\} : * \to \mathrm{Sp}$ is the constant functor at $\mathbb{S}$.

Proof. We decompose $\{\mathbb{S}\}^*$ as the composition of two equivalences,

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathcal{D}) \xrightarrow[\sim]{(\Sigma^\infty_+)^*} \mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathcal{D}) \xrightarrow[\sim]{\{*\}^*} \mathrm{Fun}(*, \mathcal{D}) \simeq \mathcal{D}$$

where the second equivalence is [Lur09b, 5.1.5.6] with $S = *$. ☐

5. THE SYMMETRIC MONOIDAL STRUCTURE ON $\mathcal{P}\mathrm{r}^{\mathrm{L}}$

Lurie [Lur17, 4.8] built a symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, analogous to the one possibly first constructed by Kelly in the 1-categorical realm [Kel05, 6.5]. Let us paraphrase the summary of results from [Gro20, 5.31]:

Proposition 5.1. *The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits a closed symmetric monoidal structure such that the following properties are satisfied:*

(1) For $\mathcal{C}, \mathcal{C}' \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$, we have

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{C}', \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L,L}}(\mathcal{C} \times \mathcal{C}', \mathcal{D})$$

for all $\mathcal{D} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$, where $\mathrm{Fun}^{\mathrm{L,L}}$ denotes functors preserving colimits separately in each variable.

(2) We have $\mathcal{C} \otimes \mathcal{C}' \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{C}')$ where $\mathrm{Fun}^{\mathrm{R}}$ denotes limit-preserving, accesible functors.[11]

(3) The ∞-category of spaces $\mathcal{S}$ is the monoidal unit.

(4) The internal hom is given by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{C}')$.

[11] This really uses the presentability hypothesis. Also, note that the adjoint functor theorem says that $\mathrm{Fun}^{\mathrm{R}}$ is given equivalently by right-adjoint functors.

Most important is (1), which says that $\otimes$ behaves as (and in fact, generalizes) the tensor product of modules over a commutative ring.

Since we have a symmetric monoidal ∞-category, we can wonder what are the commutative algebras therein.

Proposition 5.2. *The commutative algebras in the symmetric monoidal ∞-category $(\mathcal{P}r^{L}, \otimes, \mathcal{S})$ are given by closed symmetric monoidal presentable ∞-categories. Morphisms of commutative algebras correspond to (strong) symmetric monoidal functors.*

The above shouldn't be surprising if we know the 1-categorical analogue: symmetric monoidal categories are equivalently the symmetric pseudomonoids in the 2-category of categories. The closedness comes from considering left adjoint functors as the morphisms of $\mathcal{P}r^{L}$.

The following is a crucial step in our goal towards establishing the smash product:

Proposition 5.3. [Lur17, 4.8.1.23] *The functors $\mathcal{P}r^{L} \to \mathcal{P}r^{L}$ given by* $\mathrm{Sp}(-)$ *and* $\mathrm{Sp} \otimes -$ *are equivalent, together with their unit natural transformations* Σ^{∞}_{+} *and* $\mathrm{Sp} \otimes \mathrm{id}$.

Proof. By [Lur17, 1.4.2.25], the ∞-category $\mathrm{Sp}(\mathcal{C})$ is equivalent to the limit of ∞-categories[12]

$$\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*$$

where $\mathcal{C}_*$ denotes the pointed objects in $\mathcal{C}$. On the other hand, by [Lur09b, 4.8.1.21],

$$\mathcal{C}_* \simeq \mathcal{C} \otimes \mathcal{S}_* \simeq \mathrm{Fun}^{R}(\mathcal{C}^{op}, \mathcal{S}_*),$$

so we have

$$\mathrm{Sp}(\mathcal{C}) \simeq \lim_n \mathcal{C}_* \simeq \lim_n \mathrm{Fun}^{R}(\mathcal{C}^{op}, \mathcal{S}_*) \simeq \mathrm{Fun}^{R}(\mathcal{C}^{op}, \lim_n \mathcal{S}_*) \simeq \mathrm{Fun}^{R}(\mathcal{C}^{op}, \mathrm{Sp}) \simeq \mathcal{C} \otimes \mathrm{Sp}. \qquad \square$$

Since $\mathrm{Sp}(-)$ is a localization, then so is $\mathrm{Sp} \otimes -$. The unit of the localization $\mathrm{Sp}(-)$ is Σ^{∞}_{+} (Proposition 4.11). By Remark 4.4, the idempotent object associated to this smashing localization is therefore Sp via the functor $(\Sigma^{\infty}_{+})_{\mathcal{S}} : \mathcal{S} \to \mathrm{Sp}$. Under the equivalence

$$\mathrm{Fun}^{L}(\mathcal{S}, \mathcal{D}) \xrightarrow[\sim]{\{*\}^*} \mathrm{Fun}(*, \mathcal{D}) \simeq \mathcal{D}$$

which already appeared in the proof of Corollary 4.12, this functor corresponds to $\Sigma^{\infty}_{+}(*) = \mathbb{S} \in \mathrm{Sp}$, the sphere spectrum.

We now apply Corollary 4.9 to get:

Corollary 5.4. *The ∞-category of spectra* Sp *admits an essentially unique structure of a commutative algebra in $\mathcal{P}r^{L}$ with monoidal unit given by* $\mathbb{S}$.

In other words (using Proposition 5.2) the ∞-category Sp admits an essentially unique closed symmetric monoidal structure with $\mathbb{S}$ as its monoidal unit: namely, the smash product. This uniqueness statement proves that we recover the classical smash product. Indeed, if we take e.g. EKMM spectra, they are a model category whose underlying ∞-category is equivalent to Sp, and they have a closed symmetric monoidal model structure with unit the sphere spectrum, so the underlying ∞-category gets a closed symmetric monoidal ∞-structure with unit the sphere spectrum, whence from the above corollary we recover the structure we have just built.

Since we now know that Sp is an idempotent commutative algebra in $\mathcal{P}r^{L}$, we apply Proposition 4.10:

[12]The inclusion of $\mathcal{P}r^{L}$ in the ∞-category of ∞-categories preserves limits [Lur09b, 5.5.3.3], so we can take the limit in whichever of these.

Corollary 5.5. *There is an equivalence of symmetric monoidal* ∞*-categories between* $(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}, \otimes, \mathrm{Sp})$ *and* $(\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}), \otimes, \mathrm{Sp})$. *In particular, it induces an equivalence of* ∞*-categories of commutative algebras:*

$$\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}) \simeq \mathrm{CAlg}(\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})) = \mathrm{CAlg}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}).$$

As a consequence, if $\mathcal{D} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}})$*, then there is an essentially unique morphism* $\mathrm{Sp} \to \mathcal{D}$ *in* $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

In words: the ∞-category of stable presentable ∞-categories is equivalent to that of Sp-modules in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, and the ∞-category of stable presentable closed symmetric monoidal ∞-categories is equivalent to that of commutative Sp-algebras in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The final statement is a universal property for the smash product in Sp, and it says that if $\mathcal{D}$ is a presentable stable closed symmetric monoidal ∞-category, then there is an essentially unique colimit-preserving, symmetric monoidal functor $\mathrm{Sp} \to \mathcal{D}$. In other words, Sp is initial among stable presentable closed symmetric monoidal ∞-categories.

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