Introduction to stable homotopy theory
Exercise sheet n° 1

1. Prove there is an adjunction $\mathrm{Top} \underset{U}{\overset{(-)_+}{\rightleftarrows}} \mathrm{Top}_*$ between the disjoint basepoint functor and the forgetful functor. Therefore, U preserves limits.

2. Let $F : I \to \mathrm{Top}_*$ be a diagram of pointed spaces. Let $U : \mathrm{Top}_* \to \mathrm{Top}$ be the forgetful functor. Let $P : I \to \mathrm{Top}$ be the diagram with $P(i) = *$ for all $i \in I$. The inclusion of the basepoints gives rise to a morphism of diagrams $P \Rightarrow UF$. Let C be following pushout in Top:

$$\begin{array}{ccc} \mathrm{colim} P & \longrightarrow & \mathrm{colim}(UF) \\ \downarrow & & \downarrow \\ * & \longrightarrow & C \end{array}$$

Prove that the colimit of F is the space C together with the basepoint given by the arrow $* \to C$ in the diagram above.

Deduce that U preserves connected colimits, that is, colimits indexed by connected categories. A *connected category* is a category in which every two objects are connected by a zig-zag of arrows. For example: $* \leftarrow * \rightarrow *$ is connected, and $* \to * \to * \to \cdots$ is connected, so pushouts and sequential colimits are connected colimits.

3. i. Let X be a space. Prove that $\widetilde{H}_{n+1}(SX) \cong \widetilde{H}_n(X)$ for all $n \in \mathbb{N}$, where S is the unreduced suspension of X.
   ii. Let X be a well-pointed space, that is, the inclusion of the basepoint is a Hurewicz cofibration. Prove that the suspension of X is homotopy equivalent to the unreduced suspension of X.
   iii. Conclude that $\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$ if X is a well-pointed space.

4. i. Find an example proving that pushouts need not be invariant under weak homotopy equivalences. That is: construct a morphism of diagrams

$$\begin{array}{ccccc} Z & \longleftarrow & X & \longrightarrow & Y \\ \downarrow\sim & & \downarrow\sim & & \sim\downarrow \\ Z' & \longleftarrow & X' & \longrightarrow & Y' \end{array} \tag{0.1}$$

   such that the induced morphism of pushouts $P \to P'$ is not a weak homotopy equivalence. (Hint: spheres and disks.)
   ii. Let $f : X \to Y$ be a map of spaces. Its *mapping cylinder* is the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_0}\downarrow & \ulcorner & \downarrow \\ X \times I & \longrightarrow & Mf. \end{array}$$

   Give a concrete description of Mf, and prove that f factors as

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle r} & \\ & Mf & \end{array}$$

where r is a homotopy equivalence. With some work, you can also prove that i is a Hurewicz cofibration.[1]

iii. Let $Z \xleftarrow{g} X \xrightarrow{f} Y$ be a diagram of spaces. Let $M(f,g)$ be the *double mapping cylinder* of f and g, that is, the space defined as the pushout

$$\begin{array}{ccc} X & \xrightarrow{\quad i_f \quad} & Mf \\ {\scriptstyle i_g}\big\downarrow & \ulcorner & \big\downarrow \\ Mg & \longrightarrow & M(f,g). \end{array}$$

Give a concrete description of $M(f,g)$ not using Mf and Mg, and prove that it is invariant under weak homotopy equivalences, in the sense that given a diagram like (0.1), the induced map of double mapping cylinders is a weak equivalence. This justifies calling $M(f,g)$ the *homotopy pushout* of $Z \xleftarrow{g} X \xrightarrow{f} Y$.

iv. Note that $M(X \to *) \cong CX$. Therefore, the unreduced suspension SX is homeomorphic to the homotopy pushout of $* \leftarrow X \to *$.

[1] You may want to ask whether r is a Hurewicz fibration. It generally isn't, but there is a modification of Mf and of this factorization that does satisfy this.