Introduction to stable homotopy theory
Exercise sheet n° 2

Reminder: Top denotes the category of compactly-generated weakly Hausdorff spaces.

1. i. Let $X, Y \in \mathrm{Top}$. Assuming there is a bijection $\mathrm{Hom}_{\mathrm{Top}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathrm{Top}}(X, C(Y, Z))$ natural in X, Y, Z, prove there is a natural homeomorphism
$$C(X \times Y, Z) \cong C(X, C(Y, Z)).$$
   ii. Prove that the homeomorphism above restricts to a homeomorphism of based spaces
$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)).$$

2. Prove that $(-)_+ : \mathrm{Top} \to \mathrm{Top}_*$ satisfies $(X \times Y)_+ \cong X_+ \wedge Y_+$. Note that $(*)_+ \cong S^0$. This, plus some compatibility axioms, says that $(-)_+ : \mathrm{Top} \to \mathrm{Top}_*$ is a *symmetric monoidal functor*. Is $U : \mathrm{Top}_* \to \mathrm{Top}$ a symmetric monoidal functor?

3. Prove that the connected components of an H-group X are all homotopy equivalent. In particular, $\pi_1(X, x)$ does not depend on the choice of x.

4. i. (Eckmann–Hilton argument) Let G be a set and $(G, *, e)$, $(G, \circ, u)$ be two monoid structures on it, such that $\circ : G \times G \to G$ is a morphism of semigroups for the operation $*$, i.e. the following *exchange law* is satisfied:
$$(g \star g') \circ (h \star h') = (g \circ h) \star (g' \circ h')$$
for all $g, g', h, h' \in G$. Prove that $e = u$ and then that $* = \circ$. Prove also that the operation is commutative.[1]
   ii. Deduce that if A is a co-H-group and Z is an H-group, then $[A, Z]$ is an abelian group.

5. i. Let (X, x_0) be a pointed space. Define $\Omega'(X) = \{f : [0, s] \to X, f(0) = f(s) = x_0\}$, it is the *Moore loop space* of X. It is a topological space when we endow it with the initial topology with respect to the obvious map $\Omega' X \to \Omega X \times [0, \infty)$. Prove that $\Omega' X$ is a *group-like topological monoid*, that is, a topological monoid with up-to-homotopy inverses.
   ii. Prove that the natural map $\Omega X \to \Omega' X$ is a map of H-groups, and it is a homotopy equivalence.

6. If $f, g : X \to Y$ are homotopic, prove that their homotopy cofibers are homotopy equivalent.

7. Let $\mathcal{U}$ denote the category of *all* topological spaces. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots$ be a sequential diagram in $\mathcal{U}$, and let X be its colimit. Let $K \in \mathcal{U}$.
   i. Describe the natural map
$$\psi : \mathrm{colim}_i \mathrm{Hom}_{\mathcal{U}}(K, X_i) \to \mathrm{Hom}_{\mathcal{U}}(K, X)$$
and prove it is injective if the f_i are inclusions. Note that it is surjective if and only if every map $K \to X$ factors through one of the X_i.
   ii. Say that $g : A \to B$ in $\mathcal{U}$ is a *closed T_1 inclusion* if:

[1] In particular, the category of monoid objects in the category of monoids is isomorphic to the category of commutative monoids.

- It is a closed inclusion, i.e. $g(A) \subseteq B$ is closed and $g : A \to g(A)$ is a homeomorphism.
- For every $x \in B \setminus g(A)$, the set $\{x\} \subseteq B$ is closed.[2]

Prove that if the f_i are closed T_1 inclusions and K is compact, then ψ is a bijection. In other words, $\mathrm{Hom}_{\mathcal{U}}(K, -)$ preserves sequential colimits of closed T_1 inclusions when K is compact. More is true: prove that $\mathrm{colim}_i C(K, X_i) \to C(K, X)$ is a homeomorphism.

iii. In the above hypotheses, deduce that if the X_i, the f_i and K are pointed, then $\mathrm{colim}_i [K, X_i] \to [K, X]$ is a bijection. In particular, the natural map

$$\mathrm{colim}_i \pi_k(X_i) \to \pi_k(X)$$

is a bijection for all $k \geq 0$ (so, a group isomorphism for $k \geq 1$).

iv. Every Hurewicz cofibration between (weak) Hausdorff spaces is a closed T_1 inclusion. Therefore, in Top_*, homotopy groups commute with sequential colimits of (unbased) Hurewicz cofibrations.

v. Let us now work in Top. Instead of taking the ordinary colimit $\mathrm{colim}_i X_i$, which requires some point-set hypotheses for it to commute with homotopy groups, we can take its *mapping telescope*. Define it to be X_∞, the sequential colimit of the following mapping cylinders (you should make a drawing):

$$\begin{array}{ccccccccccc}
 & & Mf_0 & \xrightarrow{i_0} & M(f_1 \circ r_0) & \xrightarrow{i_1} & M(f_2 \circ r_1) & \longrightarrow & \dots & \longrightarrow & X_\infty \\
 & & \downarrow{\scriptstyle r_0} & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & & & \downarrow \\
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \longrightarrow & \dots & \longrightarrow & X
\end{array}$$

You could prove that:

- If you have a map of sequential diagrams in Top, i.e. a ladder diagram, in which all the vertical maps are weak equivalences, then the induced map on mapping telescopes is a weak equivalence. This justifies calling X_∞ the (sequential) *homotopy colimit* of the diagram.
- The canonical map $X_\infty \to X$ is a weak equivalence when the f_i are cofibrations.
- Homotopy groups commute with sequential homotopy colimits, as do ordinary homology groups.
- If X is a CW-complex, then X is the homotopy colimit of its skeleta.

[2]This is automatic if B is T_1.