Introduction to stable homotopy theory
Exercise sheet n° 5

1. i. Let M be a commutative monoid. Let $F(M)$ be $(M \times M)/_\sim$, where $(a,b) \sim (a',b')$ if there exists a $c \in M$ such tht $a + b' + c = a' + b + c$, with addition defined component-wise. Prove that $F(M)$ is an abelian group, and that the construction can be extended to a functor F : CMon $\to$ Ab which is a left adjoint to the forgetful functor U.
   ii. Let $i : M \to UF(M)$ be the unit map. Prove that $i(m) = i(m')$ if and only if there exists a $c \in M$ such that $m + c = m' + c$.
   iii. Prove that i is injective if and only if M is *cancellative*, i.e. $a + b = a' + b$ implies $a = a'$, for all $a, a', b \in M$. In this case the c in the definition of $F(M)$ and in the previous part can be taken to be 0.
   iv. Let $F'(M)$ be the abelian group defined as the quotient of the free abelian group $LU(M)$ on the set $U(M)$, quotiented by the subgroup generated by the elements $a \oplus a' - (a + a')$ where $+$ is the sum in M and $\oplus$ is the sum in $LU(M)$. Prove that F' is also a left adjoint to U : Ab $\to$ CMon, and therefore $F \cong F'$.

2. Let X be a pointed compact space. Prove that $\mathcal{E}(X) \cong \widetilde{K}(X)$.[1]

3. Let X be a finite-dimensional pointed CW-complex and F be a spectrum. The skeletal filtration on X begets a right half-plane spectral sequence with 1-st page $E_1^{p,q} = F^{p+q}(X_p/X_{p-1})$ and the differentials are the ones computing reduced cellular cohomology with coefficients in $F^q(S^0)$, so that $E_2^{p,q} \cong \widetilde{H}^p(X; F^q(S^0))$ (ordinary cohomology). It converges to $F^{p+q}(X)$:

$$E_2^{p,q} = \widetilde{H}^p(X; F^q(S^0)) \Rightarrow F^{p+q}(X).$$

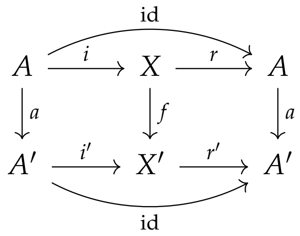
   See [Hat04, Page 537], [Sel97, 13.1.6], [Ada74, III.7], [Boa99] for details on this *Atiyah–Hirzebruch spectral sequence*. Use it to compute:
   i. $KU^*(\mathbb{C}P^n)$. More generally, compute $KU^*(X)$ where X is a finite CW-complex with no odd-degree cohomology.
   ii. $KU^*(\Sigma_g)$ where Σ_g is the real compact orientable surface of genus g.

4. Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a class of morphisms in $\mathcal{C}$. Suppose $f : X \to Y$, $g : Y \to Z$ and $g \circ f : X \to Z$ are in $\mathcal{W}$. Prove that the zig-zag $Z \xleftarrow{g} Y \xleftarrow{f} X$ is equivalent to $Z \xleftarrow{g \circ f} X$ under the equivalence relation of zig-zags in $\mathcal{C}[\mathcal{W}^{-1}]$.

5. Let $F : \mathcal{C} \to \mathcal{A}$ be a functor. Let $\mathcal{W}$ be the class of weak equivalences given by the arrows in $\mathcal{C}$ which are mapped to an isomorphism via F. Prove that $\mathcal{C}[\mathcal{W}^{-1}]$ satisfies the following closure properties, so in particular Sp[stable eq.$^{-1}$] satisfies them:
   i. The 2-out-of-3 property: If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then if two of f, g, or $g \circ f$ is a weak equivalence, so is the third one.
   ii. $\mathcal{W}$ is a wide subcategory: it is closed under composition and contains all objects and identities of $\mathcal{C}$.

[1] Recall that $\mathcal{E}(X)$ denotes the abelian group of stable equivalence classes of complex vector bundles over X.

iii. Closure under retracts: Given the following commutative diagram, if f is a weak equivalence, then so is a.



6. i. Recall that homotopy groups of spaces commute with arbitrary products.
   ii. Let X, Y be spectra and $k \in \mathbb{Z}$. Prove that $\pi_k(X \times Y) \cong \pi_k(X) \times \pi_k(Y)$ naturally; more generally, π_k preserves finite products.
   iii. Prove that the arbitrary product of Ω-spectra is an Ω-spectrum, and deduce that homotopy groups of Ω-spectra commute with infinite products.
   iv. Prove that homotopy groups of spectra may not commute with infinite products. *(Hint: consider the spectra $\mathbb{S}^{\leq i}$, which have S^n in the levels n up to i, after which they have $*$.)*

References

[Ada74] J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, 1974.

[Boa99] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.

[Hat04] Allen Hatcher. Spectral sequences. *preprint*, 2004. Available at `https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf`.

[Sel97] Paul Selick. *Introduction to homotopy theory*, volume 9 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1997.