Introduction to stable homotopy theory

Bruno Stonek

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8, 00-656 WARSZAWA, POLAND
E-mail address: bruno@stonek.com

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CHAPTER 1

Introduction

Let us start by giving a hand-wavy summary of the big lines this course will be following.

0.1. Some classical words about spectra and homology theories. One could say that stable homotopy theory started with the discovery by Freudenthal in 1937 of his "suspension theorem". One version of this theorem says that homotopy groups of spheres "stabilize", in the sense that the value of $\pi_{n+k}(S^n)$ becomes constant after a large enough n. This value is called the k-th *stable stem*, or *k-th stable homotopy group of spheres*. Whereas (unstable) homotopy groups of spheres are indexed by two integers, the stable stems only need one subindex. Only a subset of the unstable groups are stable, but there are more techniques to compute them: those of stable homotopy theory. Describing the homotopy groups of spheres is one of the big open problems in algebraic topology.

One can pass from one sphere to the next-dimensional one by means of the *suspension functor* Σ: indeed, $\Sigma S^n \cong S^{n+1}$. A more general version of the Freudenthal suspension theorem says that, for nice enough spaces X, homotopy groups stabilize after suspending enough times: these are its *stable homotopy groups*, $\pi_k^s(X)$.

More abstractly, there is a category of *spectra* Sp and a functor $\Sigma^\infty : \mathrm{Top}_* \to \mathrm{Sp}$ from the category of pointed spaces, which assigns to each pointed space X its *suspension spectrum*, which we for the moment think of as the sequence $\{X, \Sigma X, \Sigma^2 X, \dots\}$.

Spectra have homotopy groups (also negative ones). If a spectrum is of the form $\Sigma^\infty X$, then its non-negative homotopy groups are precisely the stable homotopy groups of X. If $X = S^0$, then $\Sigma^\infty S^0$ is called the *sphere spectrum* and is denoted $\mathbb{S}$: its homotopy groups are the stable stems.

Stable homotopy groups of spaces satisfy $\pi_{k+1}^s(\Sigma X) \cong \pi_k^s(X)$. Even more is true: stable homotopy theory is an extraordinary homology theory, i.e. it satisfies all the Eilenberg-Steenrod axioms of a classical homology theory in pointed spaces, except the so-called dimension axiom, the one that says that the homology groups of S^0 are trivial. Indeed, in this case they are far from trivial!

An operation on pointed spaces is said to be *stable* if it is invariant under suspension, as π_*^s above. Thus, homology theories are stable by definition. Therefore, if we are interested in the information a homology theory can give us of a given space X, in fact we can consider the spectrum $\Sigma^\infty X$: more precisely, the homology functor from pointed spaces to graded abelian groups factors via spectra.

Let us be more precise about what we mean by a "homology theory". Remember that the Eilenberg-Steenrod axioms characterize "classical homology": up to natural isomorphism, there is a unique sequence of functors from pointed spaces to abelian groups which satisfies the axioms. Removing the dimension axiom lifts this uniqueness: there are many non-isomorphic *extraordinary* homology theories on pointed spaces. They are useful in different ways. Some of the most famous extraordinary cohomology theories are topological K-theory and cobordism.

These (extraordinary) homology theories are tightly related to spectra. Indeed, any spectrum gives rise to a homology theory, and conversely, any homology theory is *represented* by a spectrum: this is called the Brown representability theorem.

0.2. The category of spectra. It is important to think not only of spectra but rather of the category they form, because indeed one should be able to define maps of spectra. It was understood early on that a "good category of spectra", or, using an alternative name, the *stable homotopy category*, should satisfy certain properties, for example, being triangulated. It took some years to settle this. Spanier and Whitehead constructed one such category in 1953, but it was then understood to be too small (it is now known as the *Spanier-Whitehead category*). Then came some other constructions, and it was finally Michael Boardman, around 1965, who gave the construction of a category widely accepted to be "the" stable homotopy category.

The struggle to set the foundations for stable homotopy theory on firm ground did not end there. There are some complications arising from the fact that to consider spectra as a 1-category is reductive. The stable homotopy category should be understood as *a homotopy category*, that is, the result of inverting weak equivalences in a category endowed with such a special class of arrows.

One example of a construction which takes some work to formulate is homotopy limits and colimits. The problem is already present unstably, in the category of spaces, so let us look at it there for simplicity. In the homotopy category of spaces, there are finite products and arbitrary coproducts, but that's it for strict (co)limits: the functor from spaces to the homotopy category does not create other (co)limits. What we would like is to make sense of *homotopy (co)limits*. For example, the set underlying the pushout of $B \xleftarrow{f} A \xrightarrow{g} C$ is the pushout of the underlying sets, which is the quotient of $B \sqcup C$ under the relation $f(a) = g(a)$. But in spaces we can do something finer: instead of setting $f(a) = g(a)$, we could add a path between $f(a)$ and $g(a)$. The upshot is that this homotopy pushout will be *homotopy invariant*: two pushout diagrams which are pointwise homotopy equivalent have equivalent homotopy colimits, which was not necessarily true for classical colimits.

The problem is then that the homotopy category of spaces, the one were we have *inverted* (weak) homotopy equivalences, is too coarse an object. We should not be turning weak equivalences into isomorphisms. We should instead be keeping track of this information, since it may be useful to build objects we are unable to build in the homotopy category, such as homotopy colimits. One such tool are *model categories*, invented by Daniel Quillen in 1967. Another such tool is ∞-*categories*, invented by Boardman and Vogt in 1973 but popularized by Joyal and Lurie in the 21st century. We can make sense of homotopy colimits both in the model category of spaces, and in the ∞-category of spaces.

The discussion above applies to spectra as well.

0.3. The smash product. In (good enough) pointed spaces, we have an additional piece of structure, which is the *smash product* $X \wedge Y$. It is a symmetric monoidal product with unit S^0, the 0-sphere. Some elementary properties of it are that $S^1 \wedge X \simeq \Sigma X$, that the disjoint basepoint functor $(-)_+ : (\mathrm{Top}, \times) \to (\mathrm{Top}_*, \wedge)$ is symmetric monoidal, and that the smash product is left adjoint to the basepoint-preserving function set (a "tensor-hom" adjunction). For example, the Künneth theorem in pointed spaces is formulated in terms of the smash product.

Moreover, this construction behaves well homotopically: e.g. we can say that the model category of spaces is a *symmetric monoidal model category* (there is a compatibility axiom between both structures hidden in that italicized term), which implies that the functor from it to its homotopy category is symmetric monoidal.

What about spectra? While it was quickly realized that defining a smash product of spectra was important for many reasons, it was not easy to do it, and it was first done only at the level of the homotopy category. John Adams' 1974 classic, the "blue book", made this construction, but it is quite messy.

Only in 1994 Elmendorf, Kriz, Mandell and May published a monograph where they constructed a symmetric monoidal model category of spectra (called *S-modules*) whose homotopy category would recover Adams' smash product. This was a much cleaner procedure. Then other monoidal models for spectra came along: symmetric spectra, orthogonal spectra, or the symmetric monoidal ∞-category of spectra, which used very different techniques to build the smash product.

The smash product is very important to do *brave new algebra* (a term coined by Waldhausen in the early 80s), also known as *higher algebra*: algebra with spectra.

0.4. The connection to algebra. Spectra rose out of algebraic topology, but they have connections to algebra as well. Lars Hesselholt likes to explain how the sphere spectrum $\mathbb{S}$ is, really, the most fundamental arithmetic gadget, more so than the integers. The integers $\mathbb{Z}$ are a bastardization (decategorification) of $\mathbb{S}$; more fundamentally, the natural numbers $\mathbb{N}$ are a decategorification of the category of finite sets, where we just remember the cardinality; they keyword here is the Barratt–Priddy–Quillen theorem, which we can paraphrase as follows: just as $\mathbb{Z}$ is the group completion of the natural numbers, $\mathbb{S}$ is the group completion of the category of finite sets and bijections, a categorification of $\mathbb{N}$.

One idea here is that spectra are like a homotopical version of abelian groups. There is a notion of topological abelian groups, but that notion is too strict: it does not catch everything we would like to have. We want a notion of a space with a product, which is not associative and commutative in the strict sense, but rather *up to coherent homotopies*: there is a path from $(ab)c$ to $a(bc)$, but also there are paths between the five $(ab)(cd)$, $(a(b(cd))$, $(a(bc))d$, $((ab)c)d$, $a((bc)d)$ and this pentagon in X can be filled, etc, etc... It turns out, by a classical theorem of Boardman-Vogt and May, that *connective* spectra (those with homotopy concentrated in non-negative degrees) are equivalent to these "homotopical abelian groups", sometimes called *grouplike E_∞-spaces*.

There is a fully faithful *Eilenberg-Mac Lane functor H* from abelian groups to spectra. With regards to the intepretation above, this amounts to seeing a strictly associative and commutative group as one which is only so up to coherent homotopy. It is also interesting to note that HA is the spectrum that gives rise to the classical homology theory of spaces $H_*(-;A)$.

Moreover, $H : (\mathrm{Ab}, \otimes) \to (\mathrm{Sp}, \wedge)$ is a symmetric monoidal functor (once we have a sufficiently high technology symmetric monoidal category of spectra!), so it preserves commutative monoids, i.e. it takes rings to *ring spectra*. A ring spectrum is, by definition, a homotopy-coherent commutative monoid in spectra. When we consider the ring spectrum HR, we are considering the classical homology begotten by R, together with the cup-product.

Now, all this looks similar to a familiar story in algebra. We can consider another fully faithful, symmetric monoidal embedding of abelian groups: namely, the functor that embeds them as chain complexes concentrated in degree zero. Just as with chain complexes we do homological algebra, with spectra we do *homotopical algebra*. But the latter is more general, because we work "over the sphere spectrum" instead of over $\mathbb{Z}$: indeed, while $\mathbb{Z}$ is the initial ring, $H\mathbb{Z}$ is no longer the initial ring spectrum, but rather the sphere spectrum $\mathbb{S}$ is.

A ring spectrum would then be the analog of a differential graded algebra over $\mathbb{Z}$. In fact, we can say more. Just as we can talk of ring spectra, we can talk of module spectra over them. There is a theorem that says that the category of $H\mathbb{Z}$-module spectra is equivalent (in a homotopical sense) to the category of (unbounded) chain complexes. Homological algebra is therefore, in an abstract sense, included in homotopical algebra / stable homotopy theory.

0.5. The multifaceted character of spectra, and further reading. So how to think of a spectrum? As a "stabilized space"? Or perhaps as a homology theory! What about a "homotopical abelian group"? Or a chain complex over a more fundamental arithmetic base?

All of these approximations have some truth to them. Our job now is to try to turn these intuitions into actual mathematical statements.

These notes will leave out many things. You should check out the sources cited in the bibliography. Also, before we start, if you need more motivation, you should check out this brilliant StackExchange post by Dylan Wilson. If you're interested in the early history of the subject, make sure to check [**May99b**].

0.6. Conventions. A "side remark" means a digressional remark that is intended as a commentary to get you interested: an invitation to read further.

The symbol $\cong$ denotes isomorphism in a given category, e.g. bijection of sets or homeomorphism of spaces. The symbol $\simeq$ denotes homotopy equivalence or weak homotopy equivalence (also known simply as weak equivalence).

When we say a space is "connected" we mean it is 0-connected, i.e. path-connected.

0.7. Acknowledgments. Many thanks to Cary Malkiewich for sharing a draft of his upcoming stable homotopy book and for patiently answering my questions via email. Some results in Sections 4.4, 4.5, 6.2 and Chapter 5 follow his book fairly closely.

CHAPTER 2

Stability phenomena in spaces

This is not a course in unstable homotopy theory, so we don't want to dwell too much on it, but there are still some things we should discuss. I want to emphasize above all some of the features that will illuminate the behavior of the category of spectra: spectra behave better, but in spaces we already have some traces of that good behavior, so it's worth it to bring them forth, especially as they are not theorems typically covered by algebraic topology courses, which I assume you have taken.

The first section highlights some structural properties of the category of pointed spaces. I have chosen to organize them according to categorical structures or properties: we will highlight the closed symmetric monoidal structure, the self-enrichment, the pre-triangulated structure. These words hide much information, and they categorize them in a way that is easier to grasp and remember (at least for the categorical-minded).

The second section presents some of the theorems that show how spaces behave very nicely once we start adding high connectivity hypotheses. By "passing to the limit", in a sense, this explains the corresponding theorems in the land of spectra, which satisfy the same conclusions without needing any connectivity hypotheses.

Finally, let me warn you that this section is missing a lot of details (since, again, spaces are not the main topic of this course). You should try to work them out by yourself, going to the references as needed. Some of the details and proofs left out are easy, but some can be rather hard.

1. The category of pointed spaces

1.1. What we really mean by *space*. The category of all topological spaces is not very well-behaved. For example, it is not cartesian closed, i.e. the functors $X \times -$ may not have right adjoints, unless X is nice enough. In other words, the compact-open topology on the sets of continuous functions is not categorically well-behaved in general. That is one reason why we should restrict to a *convenient category of spaces* (the terminology is due to Steenrod); see e.g. nLab:convenient category of topological spaces.

One such category is that of compactly generated weakly Hausdorff (CGWH) spaces. We do not want to spend time dwelling on point-set topology, so we prefer to direct the reader to Strickland's notes [**Str09**], Rezk's [**Rez18**] or Schwede's [**Sch**, A.2]. This category is cartesian closed, with function spaces Y^X. Every CW-complex is a CGWH space; every locally compact Hausdorff space is CGWH.

There is a price to pay, of course. For example, limits and colimits of topological spaces are created in the category of sets: the forgetful functor to sets has both left and right adjoints, given by the discrete and the indiscrete topologies. The forgetful functor from CGWH spaces to sets preserves limits, but does not preserve colimits. The forgetful functor from CGWH spaces to all spaces does not preserve colimits nor limits, in general. However, many common (co)limits are the same, see [**Rez18**, Section 10]. One added advantage of CGWH spaces is that they fix a problem with products and CW-complexes: the product in the category of spaces of two CW complexes may not admit a CW-structure, whereas when we take their product in the category of CGWH spaces, it does [**Hat02**, A.6].

Another possibility would be to dump topological spaces altogether[1] and to work with other models which are equivalent for the purposes of homotopy theory, e.g. simplicial sets [**Fri12**], [**GJ99**], and where we don't have to worry with point-set pathologies. But let us be classical for now, and stick with CGWH spaces.

We let Top denote the category of CGWH spaces and continuous functions, and from now on, "space" means "CGWH topological space". We let Top_* denote the category of pointed spaces and pointed continuous maps, and we let $\mathrm{Ho}(\mathrm{Top}_*)$ denote the homotopy category of pointed spaces, with pointed spaces as objects and based homotopy classes of pointed maps as morphisms. There is a functor $\mathrm{Top}_* \to \mathrm{Ho}(\mathrm{Top}_*)$, which is universal among the functors that invert based homotopy equivalences, i.e. if $\mathrm{Top}_* \to \mathcal{C}$ is a functor that sends based homotopy equivalences to isomorphisms, then it factors through $\mathrm{Ho}(\mathrm{Top}_*)$.

Much more could be said about $\mathrm{Ho}(\mathrm{Top}_*)$, especially if we illuminate it with abstract homotopy theory, e.g. model categories [**Hov99**], but we would rather not dwell on this too much.[2] A final remark: given a map between well-pointed spaces (see below for a definition), it is a based homotopy equivalence iff it is a homotopy equivalence [**May99a**, Page 46].

Sometiems we will want to restrict to $\mathrm{Ho}(CW_*)$, the homotopy category of pointed CW-complexes. By Whitehead's theorem, a map of CW-complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence, so we might as well think of $\mathrm{Ho}(CW_*)$ as having inverted the weak homotopy equivalences between pointed CW-complexes.

Let us now recall some facts about Top_* and its homotopy category.

1.2. Limits and colimits.

Proposition 2.1. *The forgetful functor* $U : \mathrm{Top}_* \to \mathrm{Top}$ *has a left adjoint* $(-)_+ : \mathrm{Top} \to \mathrm{Top}_*$ *which adds a disjoint basepoint.*

This immediately gives us information about limits in Top_*: they are created in Top under the forgetful functor. Informally, this means that you can compute the limit in Top, and the basepoint is the obvious one. For example, the product of pointed spaces is the product of spaces, with basepoint given by the product of the basepoints.

Colimits are a bit more complicated, but they can also be described [**Str11**, Page 63]. Let $F : I \to \mathrm{Top}_*$ be a diagram. After composing with the forgetful functor, we get $UF : I \to \mathrm{Top}$. Let $P : I \to \mathrm{Top}$ be the diagram with $P(i) = *$ for all $i \in I$. The inclusion of the basepoints gives rise to a morphism of diagrams $P \to UF$. Let C be following pushout of spaces:

$$\begin{array}{ccc} \mathrm{colim} P & \longrightarrow & \mathrm{colim}(UF) \\ \downarrow & & \downarrow \\ * & \longrightarrow & C \end{array}$$

Proposition 2.2. *The colimit of F is the space C together with the basepoint given by the arrow $* \to C$ in the diagram above.*

[1] Didn't Grothendieck say something like: "they were created by analysts and it shows"? Don't quote me on that one...

[2] There are different model structures on topological spaces, and they are all useful in one way or another, and it's important to compare them. The one we are implicitly considering in this chapter is the Hurewicz one, but there is also a different Quillen model structure, and a "mixed" model structure, less known, introduced by Cole and championed by May [**MP12**].

So, if F is such that if you put $*$ everywhere and then you take the colimit then you get $*$, then the colimit of F in spaces and pointed spaces coincide. This is true for countable sequences and pushouts, but not for coproducts. Coproducts in pointed spaces are called *wedge sums*, denoted $X \vee Y$: it is the pushout of spaces

$$\begin{array}{ccc} * \sqcup * & \xrightarrow{(x_0,y_0)} & X \sqcup Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \vee Y. \end{array}$$

Finally, note that spaces do not have a zero object, that is, an object which is both initial and final: indeed, the initial space is the empty set, and the final space is the one-point space. Pointed spaces form a *pointed category*, meaning they have a zero object: the one-point space is also initial in pointed spaces.

Now, as for limits and colimits in $\mathrm{Ho}(\mathrm{Top}_*)$: there aren't many. The right notion is that of *homotopy (co)limit*, we will say something about them below.

1.3. Closed symmetric monoidal structure and self-enrichment. For the theories of monoidal categories and enriched categories, see [**Kel05**].

Definition 2.3. Let $(X, x_0), (Y, y_0) \in \mathrm{Top}_*$.

(1) The *function space* $F(X, Y)$ is the closed subspace of Y^X on the basepoint-preserving maps. It is a pointed space with basepoint the constant pointed map.
(2) The *smash product* $X \wedge Y$ is the quotient of $X \times Y$ by the copy of $X \vee Y$ given by $X \times \{y_0\} \cup \{x_0\} \times Y$.

Proposition 2.4. *The category* Top_* *is closed symmetric monoidal, with internal hom given by* $F(X, Y)$*, monoidal product given by* $X \wedge Y$ *and unit given by* S^0*.*

The functor $(-)_+ : \mathrm{Top} \to \mathrm{Top}_*$ *is symmetric monoidal, where* Top *is endowed with the cartesian product.*

This proposition packages a fair amount of information, for example that the smash product operation is coherently associative up to isomorphism, or that we have natural bijections of sets

$$\mathrm{Hom}_{\mathrm{Top}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathrm{Top}_*}(X, F(Y, Z)). \tag{2.5}$$

Note that if we had not restricted the category of spaces and we had worked with all topological spaces, we would not have associativity of the smash product. See MO: Counterexample for associativity of smash product. The fact that $- \wedge Y$ has right adjoint $F(Y, -)$ implies that it commutes with colimits (e.g. with wedges). The monoidality of $(-)_+$ says, essentially, that $X_+ \wedge Y_+ \cong (X \times Y)_+$.

There is an additional piece of information: Top_* is enriched over itself, and for this enrichment $\wedge$ is the tensor and $F(-,-)$ is the cotensor (see [**Rie14**, 3.7] for this terminology). There is a fair amount of redundant information here, but something which is new in this statement which is not there in the statement "closed symmetric monoidal category" is that the bijection (2.5) can be promoted to a homeomorphism of based spaces

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)).$$

Finally, let us introduce the notation $[X, Y]$ to be the set of based homotopy equivalences: the hom in $\mathrm{Ho}(\mathrm{Top}_*)$. Note that

$$\pi_0 F(X, Y) \cong [S^0, F(X, Y)] \cong [X, Y],$$

where π_0 is the 0-th homotopy pointed set: the set of connected components, pointed at that of the basepoint.

1.4. Suspensions and loops.

Definition 2.6. If X is a pointed space, we let $CX = I \wedge X$ be the *(reduced) cone* of X; here $I = [0,1]$ is the interval space, pointed at 1. We let $\Sigma X = S^1 \wedge X$ denote the *(reduced) suspension* of X; here $S^1 = I/\partial I$ pointed at 1.

Example 2.7. For all $n \geq 0$, $\Sigma S^n \cong S^{n+1}$.

Now, dual to the above (under the smash–function space adjunction, or under *Eckmann-Hilton duality*, if you will):

Definition 2.8. If X is a pointed space, we let $PX = F(I, X)$ be the *path space* of X, and $\Omega X = F(S^1, X)$ be the *loop space* of X.

Remark 2.9.
(1) You can describe ΣX as the pushout of appropriate maps $CX \leftarrow X \rightarrow CX$, and dually ΩX as the pullback of appropriate maps $PX \rightarrow X \leftarrow PX$. Note that CX and PX are contractible.
(2) ΣX is always path-connected.
(3) Reduced homology of suspensions is easy: it is sometimes even taken as an axiom for homology in pointed spaces that $\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$. Dually[3], homotopy of loop spaces is easy: $\pi_n(\Omega X) \cong \pi_{n+1}(X)$. But homotopy of suspensions is hard (think homotopy of spheres!), and (co)homology of general loop spaces is also hard, though see [**Hat04**, 5.5, 5.17] and [**Nei10**, 4.1.5, 5.1.3] for the (co)homology of loop spaces of spheres.

As a particular case of the smash product–function space adjunction, we have the homeomorphism of pointed spaces

$$F(\Sigma X, Z) \cong F(X, \Omega Z)$$

and taking π_0 gives a bijection of pointed sets

$$[\Sigma X, Z] \cong [X, \Omega Z].$$

Note that we have $\pi_n F(X, Y) \cong [\Sigma^n X, Y]$.

Now, the space ΩX has an important binary operation: *concatenation of loops*, i.e. if I have two loops α, β, then I can run through α at twice the speed, then through β, getting $\beta \star \alpha$. This operation is not associative, but it is *homotopy associative*: there is a homotopy between the paths $(\gamma \star \beta) \star \alpha$ and $\gamma \star (\beta \star \alpha)$. Similarly, I can run α in the other direction, getting α^{-1}, but $\alpha^{-1} \star \alpha$ is only homotopical to the constant loop. This constant loop acts as a unit, up to homotopy. Note that the operation is not commutative, not even up to homotopy.

There's a name for a space with an operation which is homotopy associative, homotopy unital and with homotopy inverses: an *H-group*. You can read more about them in [**Ark11**, Chapter 2]. A proof that ΩZ is an H-group can be found in [**Ark11**, 2.3.2]. A more flexible notion which is often encountered is that of an *H-space*: this is a pointed space (X, e) with a multiplication $X \times X \rightarrow X$ such that $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity, possibly via basepoint-preserving maps.

Side remark 2.10. The loop space ΩX is more than just an H-group. It's a *grouplike A_∞-space*. To understand what this is about, here's an elementary theorem in algebra that one does not usually give much thought to. When we define a monoid (say), for associativity we only require $a(bc) = (ab)c$. We do not require $(ab)(cd) = (a(b(cd)) = (a(bc))d = ((ab)c)d = a((bc)d)$, or

[3] In the sense of "Eckmann-Hilton duality".

the equalities with a higher number of elements: these are automatic, says the theorem, which is easily proven by induction.

When we go up in complexity and we define a monoidal category, we have to be more careful: associativity is no longer strict, but up to natural isomorphism, and those natural isomorphisms (*associators*) are part of the data that needs to be specified. We also need to require the property that a so-called pentagon diagram, which specifies the above five-term equality, commutes: it does no longer come for free. However, Mac Lane's coherence theorem, which is less trivial than the classical algebra theorem above, tells us that all the higher associativities come for free, once we have this.

Now, in an A_∞-space, nothing comes for free. We need to specify all the associativities! So, saying that a space is an A_∞-space is saying a lot. Well, at least a priori, because here's another interesting theorem. Consider *topological monoids*: these are monoid objects in Top, so, a space with a strictly associative and unital multiplication. It is a very strict A_∞-space, where all the coherences can be taken to be equalities. But actually, it can be proven that any A_∞-space is weakly homotopy equivalent to a topological monoid! For example (but this is a universal example, as we will point out below), there is a space of *Moore loops* of a space X, which is an easy to describe topological monoid equivalent to ΩX; you can read about it in [**Ada78**, 2.2].[4]

This kind of result is called a "rectification result" and doesn't hold in all contexts. For example, we can also talk about E_∞-spaces: these are also coherently homotopy commutative.[5] But it is very much not true that every (nice enough) E_∞-space is homotopy equivalent to a topological commutative monoid.

Finally, another positive result: every A_∞-space and such that π_0 of it is a group ("group-like"), is equivalent to a loop space. So loop spaces are the archetypical grouplike A_∞-spaces. This is one of May's recognition theorems [**May72**].

If you want a leisurely read on A_∞-spaces, you can check [**Ada78**, 2.2]. That book takes the classical approach of Stasheff using associahedra. For a different, but equivalent, approach using "little intervals", see the introduction to Chapter 5 in [**Lur17**].

We will get some mileage out of knowing that ΩZ is an H-group.

Proposition 2.11. *If Y is an H-group, then $[X,Y]$ is a group for any X.*

Even more: a space Y is an H-group if and only if $[-,Y] : \mathrm{Ho}(\mathrm{Top}_*)^{\mathrm{op}} \to \mathrm{Set}_*$ factors through the category of groups, and Y is a homotopy commutative H-group if and only if $[-,Y]$ factors through the category of abelian groups.

Here Set_* denotes the category of pointed sets.

PROOF. The operation is defined pointwise, i.e. if $m : Y \times Y \to Y$ is the multiplication of Y and $f,g : X \to Y$, we define fg to be the map

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f\times g} Y \times Y \xrightarrow{m} Y.$$

The verifications are left to the reader; they are not hard, but can be found in [**Ark11**, 2.2.3]. □

In particular, π_0 of an H-group is a group.

There is a dual story here: every suspension ΣX is a *co-H-group*. That would be analogous to an H-group, only with all the arrows reversed: there is not a multiplcation but rather a comultiplication $\Sigma X \to \Sigma X \vee \Sigma X$ which "pinchs ΣX in the middle" (make a drawing!). A proposition dual to the above also holds [**Ark11**, 2.2.9]:

[4] A caveat is that the Moore loops are not a topological *group*, i.e. the inverses are only up to homotopy. But there are other models, a bit less explicit, as topological groups, see this MathOverflow answer.

[5] The "E" comes from "everything".

Proposition 2.12. *If A is a co-H-group, then $[A, Y]$ is a group for any A.*

Even more: a space A is a co-H-group if and only if $[A, -] : \mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{Set}_*$ factors through the category of groups, and A is a homotopy cocommutative co-H-group if and only if $[A, -]$ factors through the category of abelian groups.

Given that we have a bijection of pointed sets $[\Sigma X, Z] \cong [X, \Omega Z]$ and that both sides are groups, we would expect that isomorphism to be promoted to one of groups, and that is true [**Ark11**, 2.35]:

Proposition 2.13. *The bijection of pointed sets $[\Sigma X, Z] \cong [X, \Omega Z]$ is an isomorphism of groups.*

This gives a description of the group structure in the homotopy groups $\pi_n, n \geq 1$ of a based space, since $\pi_n(X) = [\Sigma^n S^0, X] \cong [S^0, \Omega^n X]$: we can see it as coming from the co-H-group structure on S^1, or as coming from the H-group structure on loop spaces.

We can go further:

Proposition 2.14. *If A is a co-H-group and Y is an H-group, so that $[A, Z]$ has two group operations, then these two operations are the same, and $[A, Z]$ is abelian.*

Proof. You can prove it like this: prove that one operation is a morphism for the other, i.e. if μ and m are the two multiplications and G is the group, you prove that $\mu : G \times G \to G$ is a morphism, where $G = (G, m)$ and $G \times G$ has the product group structure. This amounts to an equation called the *interchange law*, or *exchange law*. Then you play around a bit and get the result, see [**Ark11**, 2.2.12], which is valid for any group G and is called the *Eckmann-Hilton argument*. $\square$

As a corollary, $\pi_n(X)$, $n \geq 2$ is an abelian group, e.g. π_2 is of the form $[\Sigma S^0, \Omega X]$, and similarly for higher n. Also, $\Sigma^n X$ (resp. $\Omega^n Z$) is homotopy cocommutative (resp. homotopy commutative) as soon as $n \geq 2$.

To finish, note that Σ and Ω are functorial. You can see them as functors $\mathrm{Top}_* \to \mathrm{Top}_*$; you can prove they preserve homotopy equivalences, so they descend to functors $\mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{Ho}(\mathrm{Top}_*)$. You can also see Ω as a functor to H-groups, i.e. a loop map is a morphism of H-groups (defined in the obvious way); similarly for suspensions. The functors (Σ, Ω) form an adjoint pair.

1.5. (Co)fiber sequences, some homotopy (co)limits. If $f : X \to Y$ is a map of spaces, its *fiber* over $y_0 \in Y$ is the space $f^{-1}(y_0)$, and its *cofiber* is the space $Y/f(X)$. In other words, the fiber is the pullback of $X \to Y \leftarrow *$, and the cofiber is the pushout of $* \leftarrow X \to Y$. The problem with these notions, from a homotopical point of view, is that they are not homotopy invariant: consider the map $S^1 \to \mathbb{R}$ given by seeing $S^1 \subseteq \mathbb{R}^2$ and projecting onto the x-axis, and consider the map $S^1 \to *$, then compare the fibers. In other words, if you consider a square where the vertical maps are homotopy equivalences, then the induced map on fibers need not be a homotopy equivalence.

However, if f is a (Hurewicz) fibration, then the fibers *are* preserved by homotopy equivalences; if f is a (Hurewicz) cofibration, then the cofibers *are* preserved by homotopy equivalences. But you know that every map can be replaced by a fibration or a cofibration; more precisely, every map $X \to Y$ can be factored as a homotopy equivalence followed by a fibration: $X \xrightarrow{\sim} Pf \to Y$ (Pf is the *mapping path space*), or as a cofibration followed by a homotopy equivalence: $X \to Mf \xrightarrow{\sim} Y$ (Mf is the *mapping cylinder*), see e.g. [**May99a**, Chapters 5 and 6] or [**MV15**, Chapter 2]. [6] The mapping cylinder can be defined as the pushout $X \times I \leftarrow X \to Y$, and dually, the mapping path space can be defined as the pullback of $Y^I \to Y \leftarrow X$.

We then define the *homotopy fiber* of f over y_0 to be the fiber of $Pf \to Y$, and the *homotopy cofiber* of f to be the cofiber of $X \to Mf$. These are homotopy-invariant, and are also called *mapping cocone* and *mapping cone* respectively. They are examples of homotopy limits and homotopy colimits: of $X \to Y \leftarrow *$ and of $* \leftarrow X \to Y$ respectively.

An equivalent description of the homotopy cofiber of $f : X \to Y$ is as the pushout of $CX \leftarrow X \to Y$, i.e. you replace $X \to *$ by $X \to CX$ and only then take the pushout. You get a sequence of maps $X \xrightarrow{f} Y \to Cf$, where Cf is the homotopy cofiber of f. Dually, you can do this for homotopy fibers. Explicitly, the homotopy fiber of $f : X \to Y$ looks like

$$\{(x, \gamma) : x \in X, \gamma : I \to Y, \gamma(0) = y_0, \gamma(1) = f(x)\},$$

i.e. γ is a path in Y witnessing that x lies in the fiber up to homotopy.

Now, a *(homotopy) cofiber sequence* is a sequence $X \to Y \to C$ together with a homotopy equivalence $Cf \to C$ making the obvious triangle commute. This is a bit more general than a mere sequence of the form $X \xrightarrow{f} Y \to Cf$, and the added generality is sometimes useful.

We have to mention a subtlety related to basepoints.[7] To define the homotopy fiber of a map $f : X \to Y$, you need a basepoint in Y. The homotopy fiber of a based map $f : X \to Y$ has, then, the same definition as its unbased counterpart. On the other hand, when defining the homotopy cofiber of a map $f : X \to Y$, you do not need any basepoints. If you start with a based map $f : X \to Y$, then in the pushout of $CX \leftarrow X \to Y$ we can take CX to mean the reduced cone, or the unreduced cone. The pushouts are then different, but homotopy equivalent... at least if X is well-pointed:

Definition 2.15. A pointed space is *well-pointed* if the inclusion of the basepoint is a Hurewicz cofibration. More explicitly, $X \times \{0\} \cup \{x_0\} \times I$ is a retract of $X \times I$.

I admit that I may be sloppy when it comes to the well-pointed hypothesis; if you find any mistakes, let me know. Perhaps I can be excused by the fact that any pointed space is homotopy equivalent to a well-pointed one.[8]

Proposition 2.16. *If X is a well-pointed space, then the cone and the reduced cone of X are homotopy equivalent; also the mapping cylinder and the reduced mapping cylinder (defined by the same pushout, but in pointed spaces); and also the homotopy cofiber and the reduced homotopy cofiber.*

For the sake of brevity, if f is a map of based spaces and we say "homotopy cofiber", then we actually have in mind the reduced one.

Homotopy (co)limits of diagrams can be defined for spaces [**Dug08**], [**Str11**, Chapter 6], [**MV15**, Chapter 8] or for more general homotopy theories [**Rie14**, Chapters 5 & 6], [**Hir03**, Chapters 18 & 19].

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[6] More technologically, Hurewicz fibrations and cofibrations are part of a model structure on spaces, see [**MP12**, 17.1.1], and [**MP12**, 17.1.2] for the pointed counterpart, though note the subtleties in Remark 17.1.3...

[7] One could say that the core asymmetry is the fact that $\mathrm{Hom}_{\mathrm{Top}}(X, *)$ has exactly one element, so I don't need to specify such a map, whereas $\mathrm{Hom}_{\mathrm{Top}}(*, X)$ is a set isomorphic to the underlying set of X, so if I need such a map, I have to specify it. In other words, $*$ is initial but not final.

[8] See [**MP12**, 17.1.3]: well-pointed spaces are the cofibrant spaces in a certain (Hurewicz) model structure in Top_* where the weak equivalences are taken to be the homotopy equivalences, so we are talking about a cofibrant replacement functor. Explicitly, consider the space $X \vee I$ which "adds a whisker": it is well-pointed, and the map $X \vee I \to X$ which "shaves the whisker" is a homotopy equivalence.

We do not want to go into details, but we can say that homotopy colimits can be computed following the same philosophy as homotopy cofibers above, namely: given any diagram (above: $* \leftarrow X \to Y$), first replace it by one which is pointwise homotopy equivalent to it but *cofibrant*, and then take the usual colimit of that one.

To give a more fuzzy intuition: while in a strict colimit we impose equality relations, in a homotopy colimit we replace $x = y$ by a path from x to y. Look at the cofiber of a map $f : X \to Y$: we may identify it with the quotient $Y/f(X)$, i.e. we impose the relations $f(x) = *$ inside Y. In the homotopy cofiber, on the other hand, we build a new space where all the points $f(x)$ are connected to a single point via paths. A picture can be seen in [**MV15**, Figure 2.4].

For another example, the homotopy pushout of $Z \leftarrow X \to Y$ is given by the *double mapping cylinder* [**Str11**, 6.5.1]. There are some tricks that make homotopy pushouts easier to recognize, e.g. if one of the arrows is a cofibration, then the pushout computes the homotopy pushout [**Str11**, 6.4.9]. Finally, we remark that the suspension ΣX can be described as the homotopy pushout of $* \leftarrow X \to *$: here, the arrows $X \to *$ get replaced by the cofibrations $X \to CX$.

Sequential colimits. There is an example that we are going to need, that is the one given by sequences $X_1 \to X_2 \to X_3 \to \cdots$.

The following is an exercise from the exercise sheets. It is important enough to deserve appearing here, because we will make use of it in the future.

Let $\mathcal{U}$ denote the category of *all* topological spaces. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots$ be a sequential diagram in $\mathcal{U}$, and let X be its colimit. Let $K \in \mathcal{U}$.

i. Describe the natural map

$$\psi : \mathrm{colim}_i \mathrm{Hom}_{\mathcal{U}}(K, X_i) \to \mathrm{Hom}_{\mathcal{U}}(K, X)$$

and prove it is injective if the f_i are inclusions. Note that it is surjective if and only if every map $K \to X$ factors through one of the X_i.

ii. Say that $g : A \to B$ in $\mathcal{U}$ is a *closed T_1 inclusion* if:
- It is a closed inclusion, i.e. $g(A) \subseteq B$ is closed and $g : A \to g(A)$ is a homeomorphism.
- For every $x \in B \setminus g(A)$, the set $\{x\} \subseteq B$ is closed.[9]

Prove that if the f_i are closed T_1 inclusions and K is compact, then ψ is a bijection. In other words, $\mathrm{Hom}_{\mathcal{U}}(K, -)$ preserves sequential colimits of closed T_1 inclusions when K is compact. More is true: prove that $\mathrm{colim}_i C(K, X_i) \to C(K, X)$ is a homeomorphism.

iii. In the above hypotheses, deduce that if the X_i, the f_i and K are pointed, then $\mathrm{colim}_i [K, X_i] \to [K, X]$ is a bijection. In particular, the natural map

$$\mathrm{colim}_i \pi_k(X_i) \to \pi_k(X)$$

is a bijection for all $k \geq 0$ (so, a group isomorphism for $k \geq 1$).

iv. Every Hurewicz cofibration between (weak) Hausdorff spaces is a closed T_1 inclusion. Therefore, in Top_*, homotopy groups commute with sequential colimits of (unbased) Hurewicz cofibrations.

v. Let us now work in Top. Instead of taking the ordinary colimit $\mathrm{colim}_i X_i$, which requires some point-set hypotheses for it to commute with homotopy groups, we can take its *mapping telescope*. Define it to be X_∞, the sequential colimit of the following mapping cylinders

[9] This is automatic if B is T_1.

(you should make a drawing):

$$\begin{array}{ccccccccc} & & Mf_0 & \xrightarrow{i_0} & M(f_1\circ r_0) & \xrightarrow{i_1} & M(f_2\circ r_1) & \longrightarrow \dots \longrightarrow & X_\infty \\ & & \downarrow{\scriptstyle r_0} & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & \downarrow \\ X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \longrightarrow \dots \longrightarrow & X \end{array}$$

You could prove that:

- If you have a map of sequential diagrams in Top, i.e. a ladder diagram, in which all the vertical maps are weak equivalences, then the induced map on mapping telescopes is a weak equivalence. This justifies calling X_∞ the (sequential) *homotopy colimit* of the diagram.
- The canonical map $X_\infty \to X$ is a weak equivalence when the f_i are cofibrations.
- Homotopy groups commute with sequential homotopy colimits, as do ordinary homology groups.
- If X is a CW-complex, then X is the homotopy colimit of its skeleta.

Pushouts. Let D be the diagram $Z \stackrel{g}{\leftarrow} X \stackrel{f}{\rightarrow} Y$ in Top. We can also take it in Top_*, in which case all spaces should be well-pointed, and the cones, cylinders etc. should be reduced.

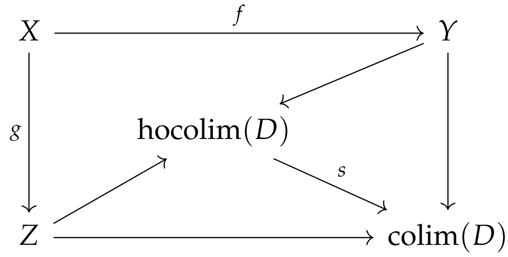
- The *homotopy pushout* of D is the double mapping cylinder of f and g, that is, the (strict) pushout

$$\begin{array}{ccc} X & \longrightarrow & Mf \\ \downarrow & \ulcorner & \downarrow \\ Mg & \longrightarrow & \mathrm{hocolim}(D). \end{array}$$

  What we did was replace the arrows f and g by the equivalent up to homotopy $X \to Mf$ and $Y \to Mg$ which have the advantage of being cofibrations.
- We don't actually need to replace both maps by cofibrations, you can prove that you can replace just one of the two and leave the other one intact, getting a homeomorphic space. For example: the homotopy pushout of $* \leftarrow X \stackrel{f}{\rightarrow} Y$ is, by definition, the pushout of $CX \leftarrow X \stackrel{f}{\rightarrow} Y$, but it is homeomorphic to the pushout of $* \leftarrow X \to Mf$, which is, by definition, the homotopy cofiber of f.
- Note that the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathrm{hocolim}(D). \end{array}$$

  is *homotopy* commutative (make a drawing!). By collapsing the cylinder to a point, we get a comparison map $s : \mathrm{hocolim}(D) \to \mathrm{colim}(D)$, and a diagram



  where the upper left part homotopy commutes, the two triangles strictly commute, and the square strictly commutes. More generally, if you start with an outer square as above which in the bottom right corner has a space P and this square is homotopy commutative, then a choice of a homotopy defines you a comparison map $s : \mathrm{hocolim}(D) \to P$ that makes the two triangles homotopy commute. This motivates the following point.

- A square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & P \end{array}$$

is *homotopy cocartesian* (or *a homotopy pushout*, note the indefinite article!) if it is homotopy commutative, and there exists a homotopy such that the induced map $s : \mathrm{hocolim}(D) \to P$ is a weak homotopy equivalence.[10]

- It is useful to know when the comparison map $s : \mathrm{hocolim}(D) \to \mathrm{colim}(D)$ is a weak homotopy equivalence. It is enough for one of f or g to be a cofibration (and in this case, the resulting map is moreover an actual homotopy equivalence). This is nice, because it means we don't need to add any of those extra cylinders, we can take the ordinary colimit and "it computes the homotopy colimit", as people say.
- Similarly, homotopy invariance holds for ordinary colimits as soon as one of the maps is a cofibration. In other words, if

$$\begin{array}{ccccc} Z & \xleftarrow{g} & X & \xrightarrow{f} & Y \\ \sim\downarrow & & \downarrow\sim & & \downarrow\sim \\ Z' & \xleftarrow[g']{} & X' & \xrightarrow[f']{} & Y' \end{array}$$

is a commutative diagram where the vertical arrows are weak homotopy equivalences and one of the arrows in each row is a cofibration, then the induced map of pushouts is a weak homotopy equivalence. Without any cofibration hypotheses, the induced map of homotopy pushouts will be a weak homotopy equivalence.

- Two important examples: the homotopy pushout of $* \leftarrow X \to *$ is ΣX in Top_* and is the unreduced suspension SX in Top. For a pointed space X, the homotopy pullback of $* \to X \leftarrow *$ is ΩX.

1.6. The pre-triangulated structure of $\mathrm{Ho}(\mathrm{Top}_*)$**.** The homotopy category of pointed spaces is not an abelian category: it is not even additive, as there need not be an abelian group operation on homsets. It is also not *triangulated*, if you know what that means (e.g. derived categories of abelian categories are triangulated). But it does admit a *pre-triangulation*: something definitely weaker. The homotopy category of spectra, in turn, will be triangulated. We shall not define the notion of pre-triangulation as it is a bit involved[11], but we are going to mention some of its more salient features for the case of pointed spaces.

A pre-triangulated category is, in particular, endowed with classes of *cofiber sequences* and *fiber sequences*. We have already defined these. They satisfy some elementary properties, such as $* \to X \xrightarrow{\mathrm{id}} X$ is a cofiber sequence, or $X \xrightarrow{\mathrm{id}} X \to *$ is a fiber sequence. Also, every map is part of a cofiber sequence and of a fiber sequence. Let us focus on cofiber sequences, but know that everything which will now be said can be dualized.

[10] Some authors may deem it wiser to include such a homotopy as part of the definition.

[11] Beware, there are at least two non-equivalent definitions of this! One of them is given in Neeman's book on triangulated categories, and it assumes that the category is additive. A non-additive definition is given in [**Hov99**, 6.5]: that's the one we have in mind.

The (strict) cofiber of a map : $A \to B$, let it be $B \to E$, satisfies by definition that the composition $A \to B \to E$ is *equal* to a constant map. The homotopy cofiber of a map $f : X \to Y$, denoted $i : Y \to C$, on the other hand, satisfies that the composition $X \to Y \to C$ is *nullhomotopic* and $Y \to C$ is universal with respect to this property. More precisely, the functor $[-, Z]$ takes $X \to Y \to C$ to an exact sequence of pointed sets[12]

$$[C, Z] \xrightarrow{i^*} [Y, Z] \xrightarrow{f^*} [X, Z]. \tag{2.17}$$

More is true: if $Y \to D$ is a map, then nullhomotopies of $X \to Y \to D$ are in 1-1 correspondence with maps $h : C \to D$ such that $Y \xrightarrow{i} C \xrightarrow{h} D$ is equal to g. Moreover, if Z happened to be a (homotopy-commutative) H-group, then this is an exact sequence of (abelian) groups.

Cofiber sequences can be continued to the right:

Proposition 2.18. *Let $X \xrightarrow{f} Y \xrightarrow{i} C_f$ be a cofiber sequence of pointed spaces. Then there is an arrow (a* connecting map*) $C_f \to \Sigma X$ such that $Y \to C_f \to \Sigma X$ is a cofiber sequence.*

PROOF. You can prove this by inspection, just using the definition of the homotopy cofiber and of suspension. Or you can be categorical and use the pasting lemma for homotopy pushouts [**Str11**, 7.23], then it follows from:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\ * & \longrightarrow & C_f & \longrightarrow & C_i \end{array}$$

□

We want to continue taking homotopy cofibers, and identify not only the spaces but also the maps. So we need to notice that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \\ {\scriptstyle\sim}\uparrow & \nearrow & \\ C_i & & \end{array}$$

See [**May99a**, 8.4] or [**MV15**, 2.4.1.9], where the mysterious negative signs are given a geometric interpretation. See also MO: Do the signs in Puppe sequences matter?.

So if we take iterated homotopy cofibers of a based map $X \to Y$ and we incorporate the equivalences mentioned just above, we get a sequence called the *Puppe sequence*, or *Barratt-Puppe sequence*:

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_f \xrightarrow{-\Sigma q} \Sigma^2 X \xrightarrow{\Sigma^2 f} \cdots \tag{2.19}$$

in which any two consecutive maps form a cofiber sequence. As a corollary of the result around (2.17), we get that if Z is a pointed space, then there is an exact sequence of pointed sets

$$\cdots \xrightarrow{(-\Sigma f)^*} [\Sigma X, Z] \xrightarrow{q^*} [C_f, Z] \xrightarrow{i^*} [Y, Z] \xrightarrow{f^*} [X, Z].$$

Note how, on the left, we eventually start mapping from suspensions and iterated suspensions: these pointed sets become groups and abelian groups, then, and this becomes an honest exact sequence of (abelian) groups. This is often summarized by saying that (2.19) is a *(homotopy)*

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[12] $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence of pointed sets if $\{b \in B : g(b) = c_0\} = \{b \in B : \exists a \in A\ b = f(a)\}$.

coxact sequence of based spaces.

You will prove in an exercise that these exact sequences encompass the long exact sequences in cohomology groups for a cofiber sequence, and the long exact sequence in homotopy groups for a fiber sequence. What about the long exact sequence in homology? It's a bit more complicated, but we can also get it.

2. Some fundamental theorems

Let us see some examples of stable phenomena. These are statements that have connectivity hypotheses and conclusions, but which become clean statements once we pass to spectra.

2.1. Freudenthal's suspension theorem and stable homotopy groups.

Definition 2.20. Let $n \geq 0$. A space X is *n-connected* if $\pi_i(X) = 0$ for $0 \leq i \leq n$. A (-1)-connected space is a non-empty space. A map $f : X \to Y$ of spaces is *n-connected* if it is an isomorphism in π_i, $0 \leq i \leq n-1$, and an epimorphism in π_n.

Therefore, the more connected a map is, the closer it is to being a weak equivalence which would be an ∞-connected map. Observe that X is n-connected iff $* \to X$ is n-connected iff $X \to *$ is $(n+1)$-connected, and $X \to Y$ is n-connected iff its homotopy fibers are $(n-1)$-connected.

Note that Σ induces a map $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ which is a group morphism as soon as $k \geq 1$.

Theorem 2.21 (Freudenthal, 1937)**.** *Let X be an n-connected pointed space, $n \geq 0$. Then the suspension map*

$$\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism for $0 \leq k \leq 2n$ and an epimorphism for $k = 2n+1$.

Another phrasing: the unit of the (Σ, Ω) adjunction, i.e. the map $X \to \Omega\Sigma X$, is $(2n+1)$-connected as soon as X is n-connected.

A more general version: If X is a pointed CW-complex and Y is a well-pointed, n-connected space, then

$$\Sigma : [X, Y] \to [\Sigma X, \Sigma Y]$$

is an isomorphism if $\dim(X) \leq 2n$*, and surjective if* $\dim(X) = 2n+1$.[13]

Another phrasing (which is also more precise): the map $\Sigma : F(X, Y) \to F(\Sigma X, \Sigma Y)$ is $(2n-t)$-connected, where $t =$ $\dim(Y)$.

PROOF. The direct proofs are involved; we would rather not go into one here. See e.g. [**Nei10**, 4.2.4], or [**Koc96**, 3.2.2], or [**Swi75**, 6.26, 15.46], or [**tD08**, 6.4.6], or [**Hat02**, 4.24], or [**May99a**, 11.2]... The choices are endless! For the more general version, see e.g. [**tD08**, 8.4.8] or [**Ark11**, 5.6.6]. □

Remark 2.22. You can shift a chain complex A to either side by an integer, getting a chain complex $A[k]$ for any $k \in \mathbb{Z}$. It satisfies that $H_{i+k}(A) \cong H_i(A[k])$. Similarly, if X is a pointed space, we have

$$\pi_k(X) \cong \begin{cases} \pi_i(\Omega^k X) & \text{if } k \geq 0 \\ \pi_i(\Sigma^{-k} X) & \text{if } k \leq 0 \text{ with restrictions on connectivity.} \end{cases}$$

In spectra, these restrictions will be lifted.

[13] Here dim refers to the dimension of a CW-complex, i.e. the largest dimension of a cell.

Corollary 2.23. *(1) If X is n-connected, then ΣX is $(n+1)$-connected. So for any X, $\Sigma^n X$ is at least $(n-1)$-connected.*

(2) If X is a finite-dimensional CW-complex, the sequence

$$[X,Y] \xrightarrow{\Sigma} [\Sigma X, \Sigma Y] \xrightarrow{\Sigma} [\Sigma^2 X, \Sigma^2 Y] \xrightarrow{\Sigma} \cdots$$

eventually stabilizes. In particular, for $k \geq 0$, the sequence

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma^2 X) \longrightarrow \cdots$$

eventually stabilizes.

Definition 2.24. Let X, Y be pointed spaces.

(1) If X is a finite-dimensional CW-complex, then the colimit of the first sequence above is denoted $\{X, Y\}$ and is called the abelian group of *stable homotopy classes of maps* from X to Y. By the corollary above, it is $[\Sigma^n X, \Sigma^n Y]$ for n sufficiently large.

(2) For $k \geq 0$, the k-th *stable homotopy group* of X is the abelian group given as the colimit

$$\pi_k^s(X) := \mathrm{colim}_n \pi_{k+n}(\Sigma^n X)$$

of the second tower above. By the corollary above, it is $\pi_{k+n}(\Sigma^n X)$ for n sufficiently large. The stable homotopy groups of spheres $\pi_k^s(S^0)$ are called the *stable stems*.

The π_k^s extend to functors $\mathrm{Top}_* \to \mathrm{Ab}$ which send based homotopy equivalences to isomorphisms, so they descend to functors $\mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{Ab}$. We will later prove that they define an extraordinary homology theory. In particular, they will have long exact sequences for cofibration sequences, something that we do not have for unstable homotopy groups, and which adds substance to the claim that stable homotopy groups are more computable than their unstable counterparts. As for $\{X, Y\}$, there is the Adams spectral sequence.

As another remark made to entice you: the group $\{X, Y\}$ will be the hom between the spectra corresponding to X and Y, under some mild conditions.

Now, for a more computational remark. See Figure 1 for a table made by Aaron Mazel-Gee showing the first stable stems. You should read it horizontal line by horizontal line. At the end of the first line you see the stable 0-stem, which is $\mathbb{Z}$, then the stable 1-stem, which is $\mathbb{Z}/2$, etc. It takes work and different techniques to do all those computations. There is a very classical one which you should know: $\pi_3(S^2) \cong \mathbb{Z}\{\eta\}$, where $\eta : S^3 \to S^2$ is the *Hopf map* (1931). Mathematicians were surprised when there turned out to be an essential map (i.e. not homotopically trivial) from a higher-dimensional sphere to a lower one; compare to the behavior in classical homology.

The next one, $\pi_4(S^3)$ is due to Serre, and is generated by the suspension of η, only it is 2-torsion (which was also surprising!). Note that Freudenthal predicts that $\pi_3(S^2) \to \pi_4(S^3)$ is a surjection but not an isomorphism, and this is tight: indeed that map is isomorphic to the quotient map $\mathbb{Z} \to \mathbb{Z}/2$. See e.g. [Hat04, 5.1] for many computations of homotopy groups of spheres with the Serre spectral sequence.

Finally, let us give a construction that will be important in the next chapter.

Definition 2.25. Let X be a based space. Define QX to be the following colimit of based spaces:

$$QX := \mathrm{colim}(\; X \longrightarrow \Omega\Sigma X \longrightarrow \Omega^2\Sigma^2 X \longrightarrow \cdots \;)$$

where the map $\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1} X$ is given by $\Omega^n \eta_{\Sigma^n X}$, where $\eta_Y : Y \to \Omega\Sigma Y$ is the unit of the (Σ, Ω) adjunction. Note that these maps $Y \to \Omega\Sigma Y$ as well as their iterated loops are all closed inclusions.

The space QX "stabilizes" X, in the following sense:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$\pi_n(S^n)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\pi_{n+1}(S^n)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_{n+2}(S^n)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_{n+3}(S^n)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}\oplus\mathbb{Z}_{12}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$
$\pi_{n+4}(S^n)$	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	0	0
$\pi_{n+5}(S^n)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
$\pi_{n+6}(S^n)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_{n+7}(S^n)$	0	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{30}$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{120}$	$\mathbb{Z}\oplus\mathbb{Z}_{120}$	$\mathbb{Z}_{240}$	$\mathbb{Z}_{240}$

FIGURE 1. The first stable stems

Proposition 2.26. *Let X be a pointed space. Then $\pi_*(QX) \cong \pi_*^s(X)$.*

PROOF. Fix $k \geq 0$. Suppose we know that π_k commutes with the colimit defining QX. The result follows after using $\pi_k(\Omega^r\Sigma^r X) \cong \pi_{k+r}(\Sigma^r X)$, and after checking that the arrows in the resulting sequence are really the arrows in the definition of $\pi_k^s(X)$.

So we need to prove that π_k commutes with that colimit. We have worked this out in Section 1.5: since the colimit defining QX is given by closed inclusions, then π_k commutes with it. □

2.2. The Blakers–Massey theorem. Some of the proofs of Freudenthal's suspension theorem cited above deduce it from a more general theorem called the Blakers–Massey theorem, or homotopy excision theorem, see e.g. [**May99a**, 11.1], or [**Hat02**, 4.23], or [**tD08**, 6.4.1]... If you look at those sources, you'll see Blakers–Massey formulated as a theorem that looks like the excision theorem for homology, only for relative homotopy groups, and only in a certain range depending on connectivity.

However, there is a more homotopical formulation that I find very appealing; I'm not sure who it is originally due to, but there is an exposition and a proof of it in [**Rez15**, 3.1]. See also these notes by Schwede, or [**MV15**, Chapter 4]. It goes like this:

Theorem 2.27 (Blakers–Massey, homotopical formulation)**.** *Consider the following homotopy push-out square:*

$$\begin{array}{ccc} Q & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longrightarrow & P. \end{array}$$

Let $R = X \times_P^h Y$ be the homotopy pullback. If f is m-connected and g is n-connected, then the canonical map $Q \to R$ is $(m+n-1)$-connected.[14]

There is also a dual version.

One can deduce the more classical statement for it, which takes a form that justifies calling it "homotopy excision theorem".

Corollary 2.28 (Classical Blakers–Massey, homotopical excision)**.** . *If $X = A \cup B$ is the union of two open subspaces such that A, B and $A \cap B$ are connected, then if $(A, A\cap B)$ is k_1-connected and $(B, A\cap B)$ is k_2-connected, for $k_1, k_2 \geq 1$, then*

$$\pi_i(A, A\cap B) \to \pi_i(X, B)$$

[14]Beware of Rezk's definition of n-connectedness: it is different by an offset of 1 from the classical one we have given here, [**Rez15**, 1.6].

is an isomorphism for $1 \leq i \leq k_1 + k_2 - 1$ *and an epimorphism for* $i = k_1 + k_2$*. Here a pair* (X, A) *is k-connected if* $\pi_i(X, A, x_0)$ *is 0 for* $1 \leq i \leq k$ *and all* x_0*, and* $\pi_0(A) \to \pi_0(X)$ *is surjective.*

Just as Freudenthal tells you how far is $X \to \Omega\Sigma X$ is from being an equivalence, relative to the connectivity of X, Blakers–Massey tells you more generally how far is a homotopy pushout square from also being a homotopy pullback square, in terms of the connectedness of the maps. More generally indeed, since Ω is a homotopy pullback, and Σ is a homotopy pushout. In spectra, we will have that homotopy pullback squares are also homotopy pushouts. You should start seeing a pattern here: these useful results about pointed spaces which are true only on a certain connectivity range, became true without assumptions in spectra.

Let's point out some particular cases of this theorem which are worthy of notice. The first one answers the question: how far is a cofiber sequence from being a fiber sequence, as well? In other words: if I have a map $X \to Y$, how far is X from being equivalent to the homotopy fiber of the homotopy cofiber of it?

Corollary 2.29. *Let* $Q \xrightarrow{g} Y \xrightarrow{c} P$ *be a cofiber sequence. Suppose that* Q *is m-connected and* g *is n-connected. Then if* F *is the homotopy fiber of* c*, the canonical map* $Q \to F$ *is* $(n + m)$*-connected.*

In spectra, we will have that cofiber sequences and fiber sequences agree, without conditions on connectivity.

Using that the homotopy pushout of $X \leftarrow X \vee Y \to Y$ is a point when X, Y are well-pointed (which is an exercise), we get:

Corollary 2.30. *Let* X, Y *be well-pointed spaces, and suppose* X *is p-connected and* Y *is q-connected. Then the canonical map* $X \vee Y \to X \times Y$ *is* $(p + q + 1)$*-connected.*

In spectra, we will have that finite coproducts and products agree, without conditions on connectivity.

Remark 2.31. A map is a 1-cube, a square of maps is a 2-cube, and you can imagine what is an n-cube for $n \geq 2$. Blakers-Massey concerns 2-cubes; its particular case Corollary 2.30 concerns 1-cubes; and there is a generalization to n-cubes, see e.g. [**MV15**, Chapters 5 & 6].

Remark 2.32. The Hurewicz theorem can also be deduced from Blakers-Massey, see [**MV15**, 4.3.2].

CHAPTER 3

Getting started with spectra

1. Cohomology theories, I

Recall the classical result that says that ordinary cohomology groups with coefficients in an abelian group G can be represented by an Eilenberg-Mac Lane space. More precisely, the functor $\widetilde{H}^n(-;G) : \mathrm{Ho}(CW_*) \to \mathrm{Ab}$, which you can construct e.g. using free abelian groups on singular chains, is representable: there exists a pointed CW-complex $K(G,n)$ such that [**Hat02**, 4.57]

$$\widetilde{H}^n(-;G) \cong [-, K(G,n)].$$

Moreover, this space has a very special homotopy type: its only non-zero homotopy group is G in degree n, but this is not the main aspect that will interest us now.

Side remark 3.1. Following up on Side remark 2.10: by Proposition 2.11, we see that $K(G,n)$ is a homotopy commutative H-group. So we are led to wonder: is it perhaps a (grouplike) E_∞-space? Actually, even more: it is a topological abelian group! In the previous side remark we said it is not true that every grouplike E_∞-space is equivalent to a topological abelian group. In fact, up to homotopy, products of $K(G,n)$ are the only ones: if you are a topological abelian group, then you are weakly equivalent to a product of $K(G,n)$. This is essentially [**Hat02**, 4K.7]. For a construction of the $K(G,n)$ as topological abelian groups, you can do the iterated bar construction, see [**Mil67**, 4.1], [**KT06**, 5.86].

What are the properties of $\widetilde{H}^n$ that make the above representability theorem true?

Theorem 3.2 (Brown representability, I)**.** *Let F be a contravariant functor from the homotopy category of pointed* connected[1] *CW-complexes to the category of pointed sets which takes coproducts to products (*wedge axiom *or* (Milnor's) additivity axiom*) and homotopy pushouts to weak pullbacks*[2] *(Mayer–Vietoris axiom)*[3]*. Then F is representable.*

PROOF. See [**AGP02**, 12.2.18], or [**Koc96**, 3.4.4], or [**Swi75**, 9.12] □

This is the original theorem of Brown, and the one that requires serious work. Below, we will see some other, easier representability theorems, which we will also call after Brown, and which follow from this one.

Now, the above concerns only each $\widetilde{H}^n$ separately. But they are linked: there is e.g. the connecting morphism in the long exact sequence, or the natural isomorphism

$$\widetilde{H}^{n+1}(\Sigma X; G) \cong \widetilde{H}^n(X, G).$$

How is this property reflected in the representing spaces? Well,

$$\widetilde{H}^{n+1}(\Sigma X) \cong [\Sigma X, K(G, n+1)] \cong [X, \Omega K(G, n+1)],$$

[1]This is important, see MO:Brown representability for non-connected spaces.

[2]A *weak pullback* is like a pullback, only that universal arrows to the pullback may not be unique.

[3]The link between this Mayer–Vietoris axiom and what is usually known as "Mayer–Vietoris" is explored in the exercises.

so as this isomorphic to $[X, K(G,n)]$ and this is a natural isomorphism in X, we get a homotopy equivalence

$$K(G,n) \xrightarrow{\sim} \Omega K(G,n+1)$$

for every $n \geq 0$, by the Yoneda lemma.

All in all, we have a sequence of pointed spaces $X_n := K(G,n)$ for $n \geq 0$ with homotopy equivalences $X_n \to \Omega X_{n+1}$. Relaxing "homotopy equivalence" to weak homotopy equivalence, we obtain the notion of an *Ω-spectrum*.

Example 3.3. Let G be an abelian group. We observed above that there is an Ω-spectrum with n-space given by an Eilenberg–Mac Lane space $K(G,n)$, which we can choose to be a CW-complex. This is the *Eilenberg–Mac Lane Ω-spectrum of G* which we will denote by HG.

The Brown representability theorem needs only the wedge axiom and the Mayer–Vietoris axiom to conclude representability. But cohomology is not characterized by these axioms: we require something stronger. If we add those stronger requirements, do we get stronger representability? The answer is yes, but first let us give the definition of a reduced generalized cohomology theory.

Definition 3.4. Let $h^n : CW_*^{\mathrm{op}} \to \mathrm{Ab}$, $n \in \mathbb{Z}$ be functors, and let $\sigma^n : h^n \Rightarrow h^{n+1} \circ \Sigma$ be natural transformations. Suppose this data satisfies the following axioms, for all n:

- (Homotopy) h^n takes homotopical maps to equal maps, i.e. h^n descends to the homotopy category $\mathrm{Ho}(CW_*)$,
- (Suspension) σ^n is a natural isomorphism,
- (Exactness) If $A \subseteq X$ is a subcomplex, then $h^n(X/A) \to h^n(Y) \to h^n(X)$ is an exact sequence.
- (Additivity) h^n takes coproducts to products.

We say that (h^n, σ^n) is a *(reduced, generalized) cohomology theory*.

Remark 3.5. We are allowing negative n here. Indeed, there are interesting cohomology theories with non-zero negative values, as we shall see. Note also that we have restricted to CW-complexes. This is not always needed, nor always a good idea: sometimes the homotopy category of well-pointed spaces is enough. In that case we need to add a weak equivalence axiom: h^n takes weak equivalences to isomorphisms, and the exactness axiom takes a cofiber sequence $A \to X \to C$ as input. The two notions are equivalent, by CW-approximation and some refinements of that, see [**May99a**, 10.7, 13.1, 14.1].

We will often omit the σ^n from the notation. Note how Mayer–Vietoris doesn't feature here: well, Mayer–Vietoris is a consequence of the above axioms [**Swi75**, 7.19]. There is a definition of a morphism of cohomology theories, which you can guess how it goes. Also, you can deduce the existence of long exact sequences in cohomology from the axioms and the Puppe sequence.

Recall the following classical result:

Theorem 3.6. *Let h^n be a cohomology theory and let G be an abelian group. There exists a unique cohomology theory up to isomorphism which satisfies the following additional axiom:*

- *(Dimension) $h^n(S^0) = 0$ for all $n \neq 0$ and $h^0(S^0) \cong G$.*

Ordinary cohomology theory in the form of e.g. singular cohomology with coefficients in G, $\widetilde{H}^n(-;G)$, satisfies these axioms.

Generalized cohomology theories can be very powerful tools, we will give examples later (perhaps the most famous and classical are topological K-theory and cobordism).

Remark 3.7. Perhaps you haven't seen these axioms but other similar axioms, the Eilenberg–Steenrod axioms, for unreduced cohomology. These are defined not in a category of pointed spaces, but in a category of pairs of spaces, i.e. they incorporate the definition of the relative homology groups $h^n(X, A)$ where A is a sub-CW-complex of A. You need not fret: a cohomology theory uniquely determines a reduced cohomology theory and viceversa (dimension axiom or not), see the following exercise.

We saw above that the representing spaces of $H^n(-;G)$ can be organized into an Ω-spectrum. Conversely and more generally:

Definition 3.8. Let E be an Ω-spectrum. For each $n \geq 0$, define

$$E^n(-) := [-, E_n] : CW_*^{\mathrm{op}} \to \mathrm{Ab},$$

and for $n < 0$ define $E^n(-) = [\Sigma^{-n}X, E_0]$.

Proposition 3.9. *The functors E^n together with the structure natural isomorphisms*

$$E^n(-) = [-, E_n] \cong [-, \Omega E_{n+1}] \cong [\Sigma -, E_{n+1}] = E^{n+1}(-) \circ \Sigma$$

define a cohomology theory.

Remark 3.10. We have defined the negative cohomology groups separately, which looks a bit odd, but this only stems from the fact that Ω-spectra are mere $\mathbb{N}$-sequence of spaces instead of $\mathbb{Z}$-sequences. This is a convention, as will be remarked in Side remark 3.17, and does not affect anything serious. If we had defined Ω-spectra as $\mathbb{Z}$-sequences of spaces, then the above would look more harmonious: we would have $E^n(-) = [-, E_n]$ for all n. Indeed,

$$[\Sigma^{-n}X, E_0] \cong [X, \Omega^{-n}E_0] \cong [X, E_{-n}].$$

You can extend an $\mathbb{N}$-based Ω-spectrum to a $\mathbb{Z}$-based one, just by defining $E_{-n} := \Omega^n E_0$ with identity transpose structure maps, and it's coherent with the above.

Theorem 3.11 (Brown's representability theorem, II)**.** *If h^* is a cohomology theory, then there exists an Ω-spectrum E and an isomorphism of cohomology theories $h^* \cong E^*$.*

PROOF. The hard part was Theorem 3.2. This part is not that hard and it's an exercise. Note how the connectedness hypothesis is gone. $\square$

We will come back to these results in more generality after we have developed some theory.

Remark 3.12. How unique is E in the previous theorem? Quite unique, but this is surprisingly subtle and requires more technology than what we have available, see math.SE:3059691.

How is the above result a "representability" result in the categorical sense? It doesn't look like one at first glance, but we will sketch below how it can be seen in this light. We are now going to work towards making the theorem above be a representability result. The plan is this: first, see the cohomology functor as landing in graded abelian groups $h^* : \mathrm{Ho}(CW_*)^{\mathrm{op}} \to \mathrm{GrAb}_{\mathbb{Z}}$. Extend this to a "cohomology theory in spectra" $h^* : \mathrm{Ho}(\mathrm{Sp})^{\mathrm{op}} \to \mathrm{GrAb}_{\mathbb{Z}}$, endow the hom-sets in $\mathrm{Ho}(\mathrm{Sp})$ with the structure of a graded abelian group, then prove that h^* is representable with respect to these.

This will be realized in Section 1, but we first need to do some work towards that goal.

2. Some elementary definitions

The Ω-spectra introduced in the previous section are special cases of the more general spectra, where we do not require the maps to be equivalences (and we also rather consider their transposes under the (Σ, Ω) adjunction in the definition):

Definition 3.13. A *spectrum* X is a sequence of pointed spaces X_n together with pointed maps $\Sigma X_n \to X_{n+1}$.

An Ω-*spectrum* X is a sequence of pointed spaces X_n together with weak homotopy equivalences $X_n \to \Omega X_{n+1}$.

Example 3.14. Consider the homeomorphisms $\Sigma S^n \xrightarrow{\cong} S^{n+1}$. This defines the *sphere spectrum* $\mathbb{S}$. This is not an Ω-spectrum. Indeed the first structure map is $S^0 \to \Omega S^1 \simeq \mathbb{Z}$.

Remark 3.15. If X is an Ω-spectrum, then there is a chain of weak homotopy equivalences

$$X_0 \xrightarrow{\sim} \Omega X_1 \xrightarrow{\sim} \Omega^2 X_2 \xrightarrow{\sim} \cdots$$

which justifies the terminology that X_0 is an *infinite loop space*. The space X_0 is the *0-space* of X and is the focus of particular attention. The space X_k is a *k-fold delooping* of X_0.

More generally:

Definition 3.16. If X is a pointed space, define the *suspension spectrum* $\Sigma^\infty X$ to be the spectrum which has $\Sigma^n X$ on level n, together with the homeomorphisms $\Sigma\Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X$.[4] Note that $\mathbb{S} = \Sigma^\infty S^0$.

If X is an unpointed space, we denote by $\Sigma^\infty_+(X)$ the spectrum $\Sigma^\infty(X_+)$, where $(-)_+$ is an added disjoint basepoint.

Side remark 3.17. This usage of the word "spectrum" is not related either to the usage in algebraic geometry (spectrum of a ring) or in functional analysis (spectrum of an operator).

Moreover: different authors may have different objects in mind when they write "spectrum". For starters, some authors allow a spectrum to consist of spaces also in negative dimension, e.g. [**Rud98**, II.1.1], though this is actually immaterial by Example 4.18. Some authors, typically Peter May and collaborators [**BMMS86**], will require the structural maps of an Ω-spectrum to be *homeomorphisms* $X_n \to \Omega X_{n+1}$: they will drop the Ω and call the more general ones as we have defined them "prespectra".

Other authors will require that the spaces X_n be CW-complexes, and that the image of $\Sigma X_n \to X_{n+1}$ be a subcomplex: these are the CW-spectra of [**Ada74**]. Yet other authors will note that indexing by the natural numbers is too rigid, and they will define *coordinate-free spectra*, where n is replaced by an n-dimensional real vector space sitting in a fixed copy of $\mathbb{R}^\infty$. The coordinate-free approach of Lewis–May [**LMSM86**] (which again favor Ω-spectra with homeomorphisms as structure maps) is the basis for the very important EKMM-spectra of [**EKMM97**], the first category of spectra with a symmetric monoidal model structure.

Around the same time, there appeared other similarly well-behaved presentations: the *symmetric spectra* of [**HSS00**], [**Sch**] and the *orthogonal spectra* [**MMSS01**]. These avoid the coordinate-free approach, but use group actions on the component spaces: symmetric groups and orthogonal groups, respectively.

Another historically important approach, because it was one of the first ways to present what we now call the stable homotopy category, was the *semisimplicial/combinatorial spectra* of Kan, based on simplicial sets rather than on topological spaces. An interesting early comparison between the earliest approaches to the stable homotopy category can be found in the introduction to [**Vog70**].

You may be starting to worry: why are there so many different versions (and the above summary does not exhaust them)? Are they not all equivalent? Well, yes and no: they are all equivalent in that they all present the same homotopy theory, for instance, they have equivalent

[4] Equivalently, we could say a suspension spectrum is a spectrum in which the structure maps are homeomorphisms.

homotopy categories: that will be *the* stable homotopy category that we will introduce below.[5] They are non-equivalent categories, though: they may have very different-looking objects and morphisms. Each has their advantages and disadvantages, but those differences are apparent later into the theory (e.g. the smash product), which is why here we stick with a simple, but for now sufficient, version.

All of the models mentioned above present the homotopy theory of spectra as a *model structure*. There are other ways of going about it, e.g. with some other presentation for $(\infty, 1)$-categories, for example quasi-categories, also known as ∞-categories, as developed by Joyal and later by Lurie [**Lur09**]. These gadgets also have homotopy categories, and there is an ∞-category of spectra whose homotopy category is the stable homotopy category [**Lur17**].

We now introduce the notion of a map of spectra, though this notion does not capture everything we want to capture: we'll explain that in Section 3.

Definition 3.18. Let X, Y be spectra. A map $f : X \to Y$ is a collection of maps of pointed spaces $f_n : X_n \to Y_n$ such that the following squares commute for all $n \geq 0$.

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\Sigma f_n} & \Sigma Y_n \\ \downarrow & & \downarrow \\ X_{n+1} & \xrightarrow[f_{n+1}]{} & Y_{n+1} \end{array}$$

There are obvious notions of composition and identity, so we have a category Sp of spectra.

Remark 3.19. You can extend Σ^∞ to a functor $\mathrm{Top}_* \to \mathrm{Sp}$, and prove that it is left adjoint to the functor $\mathrm{Sp} \to \mathrm{Top}_*$ that takes X to X_0. A word of warning: this functor is sometimes called Ω^∞, and we will also call it thusly, even if the name only makes sense when the functor is restricted to Ω-spectra (see Remark 3.15).

Let us now consider some purely categorical (not homotopical) properties of this category.

2.1. Limits and colimits. I cannot stress this enough: while sometimes these categorical constructions will be useful, they are often *not* what we would like to do when doing actual stable homotopy theory, because they may not reflect anything worthwhile homotopy-wise, cf. e.g. our discussion of homotopy colimits in the first chapter. Equivalently: these constructions are often model-dependent, in that they are going to look quite different (and perhaps be non-equivalent) in other models for spectra (as per Side remark 3.17). We only consider model-independent results to be the ones we are interested in.

Definition 3.20. The *zero spectrum* $*$ has a one-point space in each level. Alternatively, it is $\Sigma^\infty(*)$. It is a zero object in Sp.

More generally, limits and colimits are easily computed from those in Top_*, levelwise:

Proposition 3.21. *Let I be a small category, and let $X : I \to \mathrm{Sp}$ be a diagram. The colimit of X is such that $(\mathrm{colim}_i X(i))_n = \mathrm{colim}_i(X(i)_n)$. The structure maps are given by*

$$\Sigma \mathrm{colim}_i X(i)_n \cong \mathrm{colim}_i \Sigma X(i)_n \xrightarrow{\mathrm{colim}_i \rho(i)_n} \mathrm{colim}_i X(i)_{n+1}$$

where $\rho(i)_n$ is a structure map of $X(i)$. Limits are obtained similarly, only the transpose structure maps are easier to define, because Ω commutes with limits.

Therefore, Sp *is complete and cocomplete.*

[5] Though note that it took some time to reach the consensus of what, precisely, deserves to be called the stable homotopy category. See the introduction to [**Vog70**], for example. There were earlier, non-equivalent models, the best-known one being the *Spanier-Whitehead category*, which still has a place in the theory, but is not the protagonist.

Exercise 3.22. *Work out how products, coproducts, pushouts and pullbacks look like. Note that the product of Ω-spectra is an Ω-spectrum.*

2.2. Enrichment, tensoring and cotensoring over spaces. In the chapter on spaces, there was a section on the closed symmetric monoidal structure and enrichment over itself. Well, as we warned in Side remark 3.17, there is no analogue of all that for Sp as we have defined it. This is a defect of this particular model for spectra. Better-behaved categories of spectra have a closed symmetric monoidal structure behaving the way we want it to behave.

On the other hand, there is an enrichment over pointed spaces, together with a tensoring and a cotensoring (see [**Rie14**, 3.7] for details on this terminology).

Definition 3.23. If X and Y are spectra, define the pointed space $\mathrm{Map}(X,Y)$, the *mapping space from X to Y*, to be the subspace of $\prod_{n\geq 0} F(X_n, Y_n)$ given by maps which commute with the structure maps of X and Y. Categorically, $\mathrm{Map}(X,Y)$ is the equalizer in Top_*:

$$\mathrm{Map}(X,Y) \longrightarrow \prod_{n\geq 0} F(X_n, Y_n) \rightrightarrows \prod_{m\geq 0} F(\Sigma X_m, Y_{m+1})$$

where the arrows take $f_n : X_n \to Y_n$ to the object in the product given by $*$ in all degrees except for n; there, one map takes it to $\Sigma X_n \xrightarrow{\rho_n^X} X_{n+1} \xrightarrow{\Sigma f_{n+1}} \Sigma Y_{n+1}$ and the other one to $\Sigma X_n \xrightarrow{\Sigma f_n} \Sigma Y_n \xrightarrow{\rho_n^Y} \Sigma Y_{n+1}$.

Note that a point in $\mathrm{Map}(X,Y)$ is exactly a map of spectra $X \to Y$.

Definition 3.24. If X is a spectrum and K is a based space, define the *tensor* or *smash product* spectrum $X \wedge K$ by $(X\wedge K)_n = X_n \wedge K$ and structure maps

$$\Sigma(X_n \wedge K) \cong \Sigma X_n \wedge K \xrightarrow{\rho_n \wedge \mathrm{id}} X_{n+1} \wedge K$$

where ρ_n is the structure map.

Define also the *cotensor* or *function spectrum* $F(K,X)$ by $F(K,X)_n = F(K,X_n)$ with transpose structure maps

$$F(K,X_n) \to F(S^1, F(K, X_{n+1}))$$

given by

$$\rho_n \circ (\mathrm{id} \wedge \mathrm{ev}) \in \mathrm{Hom}_{\mathrm{Top}_*}(S^1 \wedge F(K,X_n) \wedge K, X_{n+1}) \cong \mathrm{Hom}_{\mathrm{Top}_*}(F(K,X_n), F(S^1, F(K, X_{n+1}))).$$

where ev denotes the evaluation map (the counit of the adjunction).

Remark 3.25. For any based space K, we have $\mathbb{S} \wedge K = \Sigma^\infty K$.

Proposition 3.26. *The category* Sp *is enriched over* Top_**, with tensor given by the smash product and cotensor given by the function spectrum. Essentially, this means that there are natural homeomorphisms of pointed spaces:*

$$F(K, \mathrm{Map}(X,Y)) \cong \mathrm{Map}(X \wedge K, Y) \cong \mathrm{Map}(X, F(K,Y)).$$

Remark 3.27. If we had defined the tensor on the other side, i.e. $K \wedge X$, then the above adjunction doesn't work. If we wanted to have the tensor like this, we should change the definition of a spectrum so as to have structure maps $X_n \wedge S^1 \to X_{n+1}$. This is the first of a series of handedness subtleties.

2.3. Other basic operations.

Definition 3.28. The *shift* of a spectrum X is the spectrum $\mathrm{sh}(X)$ defined as $\mathrm{sh}(X)_n = X_{n+1}$. The *unshift* of X is $\mathrm{sh}^{-1}(X)$, defined as $\mathrm{sh}^{-1}(X)_0 = *$, $\mathrm{sh}^{-1}(X)_n = X_{n-1}$ for $n \geq 1$. The structure maps in both of these constructions are inherited from X; for the unshift, in degree 0 it's the unique map $* \to X_0$. We can iterate these constructions, getting sh^k for all $k \in \mathbb{Z}$. They define functors $\mathrm{Sp} \to \mathrm{Sp}$.

Note that $\mathrm{sh}^n \circ \mathrm{sh}^m = \mathrm{sh}^{n+m}$ for all $n, m \geq 0$ or $n, m \leq 0$. Note also that $\mathrm{sh} \circ \mathrm{sh}^{-1} = \mathrm{id}$, but $\mathrm{sh}^{-1} \circ \mathrm{sh}$ is not the identity.

Definition 3.29. The *cylinder* of $X \in \mathrm{Sp}$ is $X \wedge I_+$, where $I = [0, 1]$ is the interval. Considering $\{0\}, \{1\} : * \to I$ gives two maps $\iota_0, \iota_1 : X \to X \wedge I_+$.

Exercise 3.30. *Prove there is an isomorphism*

$$X \wedge S^1 \cong \mathrm{cofib}(X \vee X \xrightarrow{(\iota_0, \iota_1)} X \wedge I_+)$$

where cofib *denotes the strict cofiber (i.e. the pushout of that map against the map to the point).*

We have $\mathrm{sh}(X)$ and $X \wedge S^1$. They both are "kind of suspensions" in the following sense:

Remark 3.31. The functor sh commutes with the suspension of spaces under Σ^∞, in the sense that it makes the following diagram commute.

$$\begin{array}{ccc} \mathrm{Top}_* & \xrightarrow{\Sigma} & \mathrm{Top}_* \\ {\scriptstyle \Sigma^\infty}\downarrow & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathrm{Sp} & \longrightarrow & \mathrm{Sp} \end{array} \tag{3.32}$$

The functor $- \wedge S^1$ makes the diagram commute up to natural isomorphism.

Now the question to ask is: are $\mathrm{sh}(X)$ and $X \wedge S^1$ the same? Well, they are not isomorphic. In fact, there is no natural map $X \wedge S^1 \to \mathrm{sh}(X)$ at all! The only reasonable candidate would be $\rho_n \circ \tau : X_n \wedge S^1 \to X_{n+1}$, where τ is the twist. If this defined a map of spectra, we would have a commutative diagram

$$\begin{array}{ccc} S^1 \wedge X_n \wedge S^1 & \xrightarrow{\mathrm{id} \wedge (\rho_n \circ \tau)} & S^1 \wedge X_{n+1} \\ {\scriptstyle \rho_n \wedge \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \rho_{n+1}} \\ X_{n+1} \wedge S^1 & \xrightarrow[\rho_{n+1} \circ \tau]{} & X_{n+2} \end{array}$$

But this diagram does *not* commute. An analogous argument proves that there is no natural map $\mathrm{sh}(X) \to X \wedge S^1$, either.

Side remark 3.33. In a model-categorical sense, $- \wedge S^1$ deserves to be treated as the "real suspension" as per Exercise 3.30 (which proves that $X \wedge S^1$ is the model-categorical suspension, which can be defined for any pointed model category).

As if we didn't have enough with two contenders for the "suspension" title, here's a third one:

Definition 3.34. If X is a spectrum, define its *suspension* ΣX as the spectrum having $S^1 \wedge X_n$ in degree n, and structure maps

$$\mathrm{id} \wedge \rho_n : S^1 \wedge S^1 \wedge X_n \to S^1 \wedge X_{n+1}.$$

Define the *loops* ΩX as the spectrum with $(\Omega X)_n = F(S^1, X_n)$. with transpose structure maps given by

$$F(S^1, X_n) \xrightarrow{(\widetilde{\rho}_n)_*} F(S^1, F(S^1, X_{n+1})) = \Omega F(S^1, X_{n+1}).$$

They define functors $\mathrm{Sp} \to \mathrm{Sp}$.

Exercise 3.35. *(1) Prove that* (Σ, Ω) *form an adjoint pair of functors.*

(2) Prove that Σ *also makes (3.32) commute up to natural isomorphism. But be careful: the natural isomorphism uses an associator, not the twist, which doesn't work here.*

In Chapter 5 we shall prove that ΣX, $\mathrm{sh}(X)$ and $X \wedge S^1$ are all equivalent in a homotopical sense. While we just saw that there are no direct comparison maps between $X \wedge S^1$ and $\mathrm{sh}(X)$, we can at least build a map $\Sigma X \to \mathrm{sh}(X)$. In degree n, let it be $\rho_n : S^1 \wedge X_n \to X_{n+1}$. It indeed commutes with the structure maps:

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X_n & \xrightarrow{\mathrm{id}\wedge\rho_n} & S^1 \wedge X_{n+1} \\ \downarrow{\scriptstyle \mathrm{id}\wedge\rho_n} & & \downarrow{\scriptstyle \rho_{n+1}} \\ S^1 \wedge X_n & \xrightarrow[\rho_{n+1}]{} & X_{n+2} \end{array}$$

We shall see that this map is an equivalence, in a homotopical sense.

Remark 3.36. There are two other spectra you could reasonably call the "suspension" of X. For example, you could have $S^1 \wedge X_n$ in stage n, and as structure maps you could have $S^1 \wedge S^1 \wedge X_n \xrightarrow{\tau\wedge\mathrm{id}} S^1 \wedge S^1 \wedge X_n \to X_{n+1}$. Alternatively, you could do the same on the other side.

3. An example: Topological K-theory

In this section, $[X, Y]$ will denote homotopy classes of maps between spaces without basepoint.

3.1. Vector bundles. Recall the notion of an n-plane real vector bundle: this is a map of spaces $p : E \to B$ such that for every $b \in B$ the preimage $p^{-1}(b) \subseteq E$ has the structure of a real vector space of dimension n. Moreover, the map p is locally trivial, in the sense for every $b \in B$ there is an open neighboorhood $U \subseteq B$ containing it and a homeomorphism $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$ such that the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\varphi} & U \times \mathbb{R}^n \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_0} & \\ & U & \end{array}$$

commutes, and moreover, the restrictions $\varphi : p^{-1}(b') \to \{b'\} \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ are linear isomorphisms for every $b' \in U$. We sometimes incorporate the fact that the fiber is a real vector space isomorphic to $\mathbb{R}^n$ into the notation, as $\mathbb{R}^n \to E \to B$.

In other words, n-plane real vector bundles are equivalently fiber bundles with fiber $\mathbb{R}^n$ and structure group $GL(n, \mathbb{R})$. Every vector bundle over a CW-complex (or, more generally, over a paracompact space) is a Hurewicz fibration.

I'm hoping you know about this theory at least a bit. If not, you can check [Swi75, Chapter 11], [Lee13, Chapter 10] or [DK01, Chapter 4]. Here's a couple of important examples:

Example 3.37. (1) The trivial n-plane bundle over B is given by $\mathbb{R}^n \to B \times \mathbb{R}^n \xrightarrow{\pi_0} B$.

(2) If M is a smooth manifold, then the tangent spaces at all the points $x \in M$ can be assembled into a space TM and there is a bundle $\mathbb{R}^n \to TM \to M$, the *tangent bundle*.

(3) There is a line bundle $\mathbb{R} \to E \to S^1$ called the *Möbius bundle*, where E is a space that is a non-compact version of the Möbius strip.

If we just say "vector bundle", then this means that we are not fixing the dimension, and thus, if the base is not connected, then the fibers over different connected components may have different dimensions.

It's not hard to define maps of vector bundles and isomorphism classes of vector bundles. Explicitly, an isomorphism of vector bundles from $p_1 : E_1 \to B$ to $p_2 : E_2 \to B$ is a homeomorphism $f : E_1 \to E_2$ such that $p_2 \circ f = p_1$ and the induced map on fibers $f : p_1^{-1}(b) \to p_2^{-1}(b)$ is a linear isomorphism.

Vector bundles can be pulled back along maps to the base. Note that a vector bundle $\mathbb{R}^n \to E \to B$ is trivial if and only if there exist sections $s_i : B \to E$, $i = 1, \dots, n$ such that $\{s_1(b), \cdots, s_n(b)\}$ are linearly independent for all $b \in B$.

There is a construction called the *Whitney sum* of vector bundles: if $E_1 \to B$, $E_2 \to B$ are vector bundles over the same base, then first you can construct its product, which is the obvious vector bundle $E_1 \times E_2 \to B \times B$. Pulling it back along the diagonal $B \to B \times B$ gives the *Whitney sum*, a vector bundle denoted $E_1 \oplus E_2 \to B$ whose fibers are given by the direct sum of the fibers of the two bundles.

Importantly, recall that vector bundles have classifying maps. In a word, the set of equivalence classes of n-plane real vector bundles over a paracompact B is in bijection with $[B, BO(n)]$, where $BO(n)$ is the classifying space of $O(n)$, the topological group of linear isometries of $\mathbb{R}^n$ [**AGP02**, 8.5.13]. To pass from one to the other: there exists a universal n-plane vector bundle over $BO(n)$, in the sense that for any vector bundle ξ over B, there exists a map $B \to BO(n)$, such that pullback of the universal bundle along this map gives you ξ, up to equivalence. A concrete construction of this universal vector bundle is given by the tautological bundle over the Grassmannian $G_n(\mathbb{R}^\infty) \simeq BO(n)$; alternatively, you can take your preferred model for $EO(n) \to BO(n)$, the universal principal $O(n)$-bundle, and change the fiber to $\mathbb{R}^n$, getting the universal n-plane vector bundle $EO(n) \times_{O(n)} \mathbb{R}^n \to BO(n)$.

If you are wondering about why $O(n)$ precisely, think of this: it would be more natural to consider $GL(n, \mathbb{R})$ (all linear isomorphisms), but by the Gram–Schmidt process, $GL(n, \mathbb{R})$ is homotopy equivalent to $O(n)$, see e.g. [**Swi75**, 11.44].

We can do all of the above for complex vector bundles. In that case, $U(n)$ takes the place of $O(n)$, and "dimension" means complex dimension. We could also do quaternionic vector bundles, in which case $Sp(n)$ steps in (this is the symplectic group).

Remark 3.38. We have barely said anything about what B means here. This construction, called the classifying space or the bar construction, is quite fundamental. It takes a topological group into a pointed connected space BG satisfying that G is weakly homotopy equivalent to ΩBG: thus, a topological group G admits a "connected delooping" BG, and indeed the adjunction (B, Ω) between topological groups and pointed connected spaces is such that both the unit and the counit are weak homotopy equivalences. Thus, these two categories are "equivalent" in a certain homotopical sense. Note that $\pi_{i+1}(BG) \cong \pi_i(G)$.

The space BG is also characterized by a property similar to the above: it is a space such that $[X, BG]$ is in bijection with the isomorphism classes of principal G-bundles. See [**Swi75**, 11.33] for an indirect construction via Brown's representation theorem; see [**Rud98**, IV.1.62ff.] for a more direct algebraic construction which has the added advantage of working for topological monoids.

If G is discrete, then BG is merely a $K(G, 1)$, but when the topology is more interesting, BG is richer. For example, its cohomology is closely related to characteristic classes, see e.g. [**Swi75**, 16.10] for a computation of some of these. As for the homotopy groups, we'll talk about the ones of BU below.

Side remark 3.39. The bijection between $[X, BO(n)]$ and isomorphism classes of real n-plane vector bundles is one of many instances of this dichotomy between the "fibrational" approach vs. the "functional" (or "functorial") approach. I prefer to think of them as vertical vs. horizontal, following Grothendieck's wisdom that bundles and fibrations should be written vertically.

Another instance is in 2-category theory: Grothendieck fibrations over a category $\mathcal{B}$ are equivalent to contravariant pseudofunctors from $\mathcal{B}$ to Cat, and the equivalence in one direction is given by pullback over a universal Grothendieck fibration. One can replace Cat by the 2-category of groupoids, or even with the category of sets. Going in the other direction, one can generalize it to ∞-category theory and this is the backbone for a lot of the theory in [**Lur09**] and [**Lur17**]: it's hard to overstress its importance.

In the case of functors to sets, an important example is given by the following. It was realized by Segal in [**Seg74**] that the category of commutative monoids is equivalent to the category of functors $\mathrm{Fin}_* \to \mathrm{Set}$ satisfying a couple of conditions. Passing to the fibrational world, this is equivalent to discrete fibrations over Fin_*. This is a very useful realization that helps put into motion the formalization of the concept of E_∞-monoids, an object that already appeared in Side remark 2.10. This is valid not only for monoids but for all sorts of algebraic structures, and it helps e.g. in formulating what is a symmetric monoidal ∞-category – see [**Lur17**].

Finally, another instance of the fibrational vs. functorial approach can be found in the theory of covering spaces: covering spaces over B are equivalent to locally constant sheaves over B (which can be described as functors $\Pi_1(B) \to \mathrm{Set}$ from the fundamental groupoid); more generaly, étalé spaces are equivalent to sheaves over B; more generally, $\mathrm{Top}_{/B}$ is equivalent to $\mathrm{Fun}(\mathcal{O}(B)^{\mathrm{op}}, \mathrm{Set})$, where $\mathcal{O}$ denotes the opens. This is, I think, less strictly analogous, since passage from one of the sides to the other is not given by pullback over a universal map, and we have maps over B on one side but functors out of $\mathcal{O}(B)$ on the other side.

3.2. The K-group. Sources for further reading: [**Mit11**], [**Swi75**, Chapter 11], [**Hus94**], [**Hat17**].

Let X be a space. If we want to extract information from it algebraically (i.e. do algebraic topology, literally!), we can look at its homology groups or its homotopy groups. The former is roughly about linearly mapping simplices into X, the latter is roughly about mapping spheres into X. Now, we'll take another approach: we'll look at the complex vector bundles that lie over X. We could also do it for real or symplectic vector bundles.

Definition 3.40. We let $\mathrm{Vect}(X)$ denote the set of isomorphism classes of complex vector bundles over a space X.

Now, you can prove that $\mathrm{Vect}(X)$ has a commutative monoid structure: $(\mathrm{Vect}(X), \oplus, \epsilon^0)$. Here $\oplus$ is the Whitney sum, and for every $j \geq 0$, we let ϵ^j denote the trivial j-dimensional bundle over X.

Commutative monoids are a bit unwieldy compared to abelian groups. So we want to add an inverse. We can do it "by force": one way is to take the free abelian group, and then force the new $+$ to coincide with the old $\oplus$ by taking a quotient of the objects of the form $m + n - m \oplus n$. This is a description of the left adjoint to the inclusion of abelian groups in commutative monoids. It is called *Grothendieck construction*, but beware, there are several different constructions with the same name. It's nothing too sophisticated: think of how $\mathbb{Z}$ is constructed from $\mathbb{N}$. You can represent every element of $\mathbb{Z}$ by a pair $(n, m) \in \mathbb{N} \times \mathbb{N}$, to be thought of as $n - m$. For a general monoid is sligthly more complicated (see remark below), we will look at it in the exercises.

Definition 3.41. The K-group of X, denoted $K(X)$, is the Grothendieck construction of $\text{Vect}(X)$. An element of $\text{Vect}(X)$ is called a *virtual vector bundle* over X.

If we denote by $[\xi]$ the image of ξ under the canonical map $\text{Vect}(X) \to K(X)$, then any virtual vector bundle can be written as $[\xi] - [\eta]$ for some vector bundles ξ and η.

Remark 3.42. One has to be careful, though, because e.g. $[\xi] - [\eta] = 0$ in $K(X)$ does not mean ξ and η are isomorphic vector bundles, but rather that there exists a ζ such that $\xi \oplus \zeta \cong \eta \oplus \zeta$. The reason for this discrepancy with the case where $\mathbb{Z}$ is built from $\mathbb{N}$ is that $\mathbb{N}$ is a cancellative monoid (i.e. $a + b = a + c$ implies $b = c$) which is not true for $\text{Vect}(X)$.

Using pullbacks, Vect can be extended to a contravariant functor from the category of spaces to the category of commutative monoids, and thus K is extended to a functor $K : \text{Top}^{\text{op}} \to \text{Ab}$. Moreover, you can prove that K takes homotopies to equalities.

Side remark 3.43. What would happen if we did not take isomorphism classes in the definition of Vect? Recall Side remark 2.10. We define $\pi_1(X)$ as homotopy classes of based maps $S^1 \to X$, whereas we get more information by looking at the whole space of based maps $S^1 \to X$, which is not only an abelian group up to homotopy, which provides $\pi_1(X)$ with its abelian group structure, but it's even an A_∞-space.

We should be wondering about something similar here, as we should do each time that we see some set of isomorphism classes. And indeed, we could play a similar game, but it would again require technology that is unavailable to us at this stage. I can still sketch it. I learned this from Denis Nardin, and I don't think it has been written in detail anywhere, though a similar approach to algebraic K-theory has been presented in [**Nik17**], see also [**GGN15**].

Consider $\mathcal{V}(X)$, the *category* of complex vector bundles over X. By considering it as a category, we have hom-sets of vector bundle maps $E \to E'$. This is a subset of the set $\text{Hom}_{\text{Top}}(E, E')$, which is a topological space with the compact-open topology, so we can view $\text{Hom}_{\mathcal{V}(X)}(E, E')$ as a (sub)space. Since composition is continuous, this provides $\mathcal{V}(X)$ with the structure of a category enriched over Top, also known as a topological category. Moreover, $\mathcal{V}(X)$ has a symmetric monoidal structure given by the Whitney sum, and it is compatible with the enrichment.

Now, we can restrict to the underlying groupoid $\mathcal{V}^{\sim}(X)$ (also known as the *core*): this means simply restrict to the subcategory with all objects but only isomorphisms as maps. It is a symmetric monoidal topological groupoid, and this is already a higher version of $\text{Vect}(X)$: instead of taking isomorphism classes, we are taking the isomorphisms and considering them with higher structure, namely the topological enrichment; the commutative monoid structure of $\text{Vect}(X)$ is generalized as a symmetric monoidal structure.

Now, we would like to add homotopy inverses to the symmetric monoidal structure $\oplus$, to get a higher version of $K(X)$. The standard thing to do is to first take nerves, which takes us out of topological 1-categories and into ∞-categories, which are more convenient. The traditional nerve functor takes a category and gives you a simplicial set; if your category was a groupoid, it gives you a Kan complex. More elaborately, if you have a topological category, then you have a homotopy-coherent nerve functor N^h which gives you a quasicategory (an ∞-category). If your topological category was a topological groupoid, then you get a Kan complex: a "space". Finally, if you have a symmetric monoidal topological groupoid, then its homotopy-coherent nerve is an E_∞-space. And there is a group-completion functor from E_∞-spaces to grouplike E_∞-spaces, which, as we announced in the introduction, are equivalent to the category of connective spectra, but we shall not have enough time to prove that.

In conclusion, the higher version of the abelian group $K(X)$ would be a grouplike E_∞-space, or, equivalently, a connective spectrum. The higher version of the functor $K(-)$ would be the composition

$$\mathrm{Top}^{\mathrm{op}} \xrightarrow{\mathcal{V}} \mathrm{SMTopCat} \xrightarrow{(-)^{\sim}} \mathrm{SMTopGrpd} \xrightarrow{N^h} E_\infty(\mathcal{S}) \xrightarrow{(-)^{\mathrm{grp}}} E_\infty^{\mathrm{grp}}(\mathcal{S}) \simeq \mathrm{Sp}^{\mathrm{cn}},$$

and, starting with a finite CW-complex X, we can identify the resulting connective spectrum as $F(X_+, ku)$ where ku is the connective K-theory spectrum, which we haven't yet introduced. The precise relation between $K(X)$ and $F(X, ku)$ is that $\pi_0 F(X_+, ku) \cong K(X)$. So $K(X)$ is $ku^0(X_+)$: the 0-th cohomology group for a certain spectrum ku... we're getting ahead of ourselves, but bear with me.

The above gives a possible construction of ku: do that sequence of functors starting from the one-point space. [6] In other words, start with the symmetric monoidal topological category of finite dimensional complex vector spaces, then take its core, its homotopy-coherent nerve, and then group-complete. Note that, before group-completion, the E_∞-space is $\coprod_{n\geq 0} BGL(n,\mathbb{C}) \simeq \coprod_{n\geq 0} BU(n)$. This follows from the definition of the nerve, since the isomorphisms of vector spaces are given by all the $GL(n,\mathbb{C})$. The reason its group completion is $BU \times \mathbb{Z}$ is the classical group-completion theorem of McDuff–Segal, a modern version of which can be found in [**Nik17**, Example 8].

Finally, another important realization: from the above description, we can see how topological K-theory is linked to (direct sum) algebraic K-theory. Let R be a ring. Consider the *discrete* symmetric monoidal category of finitely generated, projective R-modules. Then doing the above procedure defines an E_∞-group (or a connective spectrum) $K(R)$, its algebraic K-theory. If R has invariant basis number (e.g. R is commutative), then the intermediate E_∞-monoid is $\coprod_{n\geq 0} BGL(n,R)$... but be careful! That $GL(n,R)$ there is *discrete*, since we started with a discrete category. So it's not true that ku is the same as $K(\mathbb{C})$. While the homotopy groups of the former are easily described (we'll come back to that below), the ones of the latter, denoted $K_i(\mathbb{C})$, are much harder, and I don't think they're fully described yet. See math.SE:higher K-theory of complex numbers for a summary.

The above highlights how both topological K-theory and algebraic K-theory are two instances of the same machine, but they're also more directly related, for example via the classical Serre–Swan theorem, which says that the topological K-theory group of a compact Hausdorff space X is isomorphic to the algebraic K-theory of its ring of complex functions $C(X,\mathbb{C})$.

If X is a pointed space, there is a function $d : \mathrm{Vect}(X) \to \mathbb{Z}$ which takes a vector bundle to the dimension of its fiber over the basepoint. This is a map of commutative monoids, so it induces a homomorphism of abelian groups $d : K(X) \to \mathbb{Z}$, the *virtual dimension*. But be careful! This is sloppy terminology, because it's only about the dimension over the basepoint: if the space is not connected, the dimension over other points could vary. So we should rather say "virtual dimension over the basepoint".

Note:

Example 3.44. The map $d : K(*) \to \mathbb{Z}$ is an isomorphism, since $\mathrm{Vect}(*) \cong \mathbb{N}$. For a general X, we can therefore identify d with the map $K(X) \to K(*)$ induced by the inclusion of the basepoint $* \to X$.

Definition 3.45. For a pointed space X, define $\widetilde{K}(X)$ to be the kernel of $d : K(X) \to \mathbb{Z}$. It consists of virtual vector bundles of virtual dimension zero, or, alternatively, it consists of differences of classes of vector bundles over X whose fiber over the basepoint has the same dimension.

[6] A more hands-on construction of ku that uses the same principles can be found in [**Sch**, 1.20].

Note that d has a section, induced by the map $X \to *$, so we have a split short exact sequence

$$0 \longrightarrow \widetilde{K}(X) \longrightarrow K(X) \underset{\longleftarrow}{\overset{d}{\longrightarrow}} \mathbb{Z} \longrightarrow 0.$$

In particular, $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$.

Example 3.46. (1) $\widetilde{K}(*) = 0$.

(2) $\mathrm{Vect}(S^0) \cong \mathbb{N} \oplus \mathbb{N}$, so $K(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $\widetilde{K}(S^0) \cong \mathbb{Z}$.

(3) $\widetilde{K}(X_+) \cong K(X)$.

We can give a more direct interpretation of $\widetilde{K}(X)$ using the following definition.

Definition 3.47. Let ξ, η be two vector bundles over a space X. We say that ξ and η are *stably equivalent* if there exist $n, m \geq 0$ such that $\xi \oplus \epsilon^n \cong \eta \oplus \epsilon^m$, i.e. they become equivalent after summing with large enough trivial bundles on both sides.

Stable equivalence is an equivalence relation; we let $\mathcal{E}(X)$ denote the set of equivalences classes. We denote by $\{\xi\}$ a stable equivalence class.

We can define $\oplus$ in $\mathcal{E}(X)$ by $\{\xi\} \oplus \{\eta\} = \{\xi \oplus \eta\}$. This is a well-defined operation, associative and commutative, and it has $\{\epsilon^n\}$ as its neutral element, for any $n \geq 0$, so this is a commutative monoid. It also has inverses, at least when X is compact:

Proposition 3.48. *If ξ is a vector bundle over a compact space X[7] then there exists an $n \geq 0$ and a vector bundle η over X such that $\xi \oplus \eta \cong \epsilon^n$. Therefore, $\mathcal{E}(X)$ is an abelian group.*

PROOF. See e.g. [Kar08, I.6.5]. This is a foundational result. Essentially, it goes like this: take an open cover of X such that ξ restricted to those opens is trivial. Since X is compact, we can assume the cover is finite. Now, an n-plane vector bundle is trivial iff there exist n sections of it which are linearly independent. Using a partition of unity, you can extend these to a whole lot of sections of ξ which are generators, defining a map of bundles $f : \epsilon^N \to \xi$ which is surjective over every $x \in X$, and in fact f has a left inverse g (a consequence of the fact that every short exact sequence over $\mathbb{R}$ splits). Define η to be the kernel of $g \circ f$. □

Corollary 3.49. *Any virtual vector bundle over a compact space X is of the form $[\zeta] - [\epsilon^n]$ for some $n \geq 0$.*

PROOF. Take a virtual vector bundle $[\xi] - [\eta]$. Take $n \geq 0$ and β such that $\eta \oplus \beta \cong \epsilon^n$. Therefore,

$$[\xi] - [\eta] = [\xi] + [\beta] - [\epsilon^n] = [\zeta] - [\epsilon^n]$$

for $\zeta = \xi \oplus \beta$. □

Proposition 3.50. *Let X be a pointed compact space. Then $\mathcal{E}(X) \cong \widetilde{K}(X)$.*

PROOF. Let $\mathrm{Vect}(X) \to \mathcal{E}(X)$ be the map sending $[\xi]$ to $\{\xi\}$. It's a morphism of commutative monoids, so it defines a homomorphism $K(X) \to \mathcal{E}(X)$. Exercise: prove that when restricted to $\widetilde{K}(X)$, it's an isomorphism. □

In particular, if X is pointed compact we can describe the isomorphism $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$ as

$$[\xi] - [\epsilon^n] \mapsto (\{\xi\}, d(\xi) - n),$$

identifying $\widetilde{K}(X)$ with $\mathcal{E}(X)$ as is standard.

[7] Or a finite-dimensional CW-complex [Swi75, 11.55], both hypotheses work but neither contains the other. For simplicity, we will stick to compactness.

Example 3.51. This is a real example, not complex, for simplicity. Let M be an n-dimensional smooth manifold. Let $i_r : M \to \mathbb{R}^{n+r}$ and $i_s : M \to \mathbb{R}^{n+s}$ be embeddings into Euclidean space (they exist by Whitney's embedding theorem). There's a theorem that tells you that they are equivalent, once you go farther into a big enough $\mathbb{R}^N$.

Now, consider the normal bundles defined by the embeddings: $\mathbb{R}^r \to N(i_r) \xrightarrow{\nu i_r} M$ and $\mathbb{R}^s \to N(i_s) \xrightarrow{\nu i_s} M$. As a consequence of the above theorem, $\nu i_r \oplus \epsilon^{N-r} \cong \nu i_s \oplus \epsilon^{N-s}$.

If $r = s$ and the embeddings are not equivalent, this is an example of two bundles of the same dimension which are not equivalent but are stably equivalent.

The stable equivalence class of any normal bundle to M is called the *stable normal bundle*. Contrary to its unstable counterparts, it does not depend on the choice of embedding.

Example 3.52. Let $\mathbb{R}^n \to TM \to M$ be the tangent bundle to a manifold, let $M \to \mathbb{R}^{n+k}$ be an embedding, and let $\mathbb{R}^k \to NM \to M$ be its normal bundle. Then $TM \oplus NM \cong \epsilon^{n+k}$. So here's an example of a bundle which is not trivial but is stably trivial. Consider the tangent bundle $TS^2 \to S^2$. It is not trivial: this is the "hairy ball theorem". Consider an embedding $S^2 \to \mathbb{R}^3$ and its associated normal bundle: it is trivial (the outward-pointing unit normal vector is a nowhere-vanishing section), and the sum of the two is isomorphic to ϵ^3.

3.3. Homotopical interpretation. Let BU denote the colimit in Top_* of the maps $BU(n) \to BU(n+1)$ induced by mapping an isometry f to $f \oplus \mathrm{id}_{\mathbb{C}}$; the basepoints are the identities. One can prove that Whitney sum endows BU with the structure of a homotopy-commutative H-group, [**Swi75**, 11.58]. Therefore, $[X, BU]$ is an abelian group for all X. Similarly, since $\mathbb{Z}$ is an abelian group, then it is in particular a discrete homotopy-commutative H-group, hence $BU \times \mathbb{Z}$ is so too, and $[X, BU \times \mathbb{Z}]$ is an abelian group for all X.

Proposition 3.53. *If X is a pointed connected compact space, then there is an isomorphism of abelian groups*

$$\widetilde{K}(X) \cong [X, BU]$$

natural in X.

The contravariant functor K from the homotopy category of compact spaces to abelian groups is representable by $BU \times \mathbb{Z}$, i.e. there is an isomorphism of abelian groups

$$K(X) \cong [X, BU \times \mathbb{Z}]$$

which is natural in X.

PROOF. We define a map $T : [X, BU] \to \widetilde{K}(X)$. Since X is compact, then a map $f : X \to BU$ factors through a map to some $BU(n)$, by the results in Section 1.5. This is a classifying map to some vector bundle ξ over X, so we define $T[f] = \{\xi\}$; you need to check this is well-defined. Since any vector bundle has a classifying map, T is surjective, and it's not hard to see it's injective. See [**AGP02**, 9.4.9] for a full proof.

For $K(X)$: if X is connected, this follows immediately from the splitting $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$, since then $\mathbb{Z} \cong [X, \mathbb{Z}]$. For non-connected spaces, a bit more work is needed, see [**AGP02**, 9.4.9]. □

Remark 3.54. The above gives us a purely homotopical criterion for two vector bundles to be stably equivalent. Namely, if ξ and η are vector bundles over a pointed connected compact X, then they are stably equivalent if and only if their classifying maps $X \to BU(n)$ and $X \to BU(m)$ are such that, when composed with $BU(n) \to BU$ and $BU(m) \to BU$, the resulting maps $X \to BU$ are homotopic.

For example (in the real case), given a manifold M and two embeddings into an Euclidean space as in Example 3.51, if we take their classifying maps $M \to BO(r)$ and $M \to BO(s)$, then we have the resulting maps $M \to BO$ and they are homotopy equivalent.

Remark 3.55. Let us denote by $[X, Y]_*$ the set of pointed homotopy classes of pointed maps. We have that

$$K(X) \cong [X_+, BU \times \mathbb{Z}]_*$$

if X is compact, using the adjunction $\mathrm{Ho}(\mathrm{Top}) \underset{\text{forget}}{\overset{(-)_+}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_*)$. From this, we can deduce that

$$\widetilde{K}(X) \cong [X, BU \times \mathbb{Z}]_* \tag{3.56}$$

if X is compact and well-pointed. Indeed, since (X, x_0) is well-pointed, taking the strict cofiber of $S^0 = \{*\}_+ \xrightarrow{\{x_0\}_+} X_+$ gives a cofiber sequence $S^0 \to X_+ \to X$. Applying $[-, BU \times \mathbb{Z}]_*$ to it gives an exact sequence of abelian groups (recall the results from Section 1.6):

$$[X, BU \times \mathbb{Z}]_* \to [X_+, BU \times \mathbb{Z}]_* \to [S^0, BU \times \mathbb{Z}]_*.$$

Now, the second map can be identified with $d : K(X) \to \mathbb{Z}$. Moreover, the first map is injective, since the previous term in the Puppe sequence is $[S^1, BU \times \mathbb{Z}]_* = \pi_1(BU) = 0$, so $[X, BU \times \mathbb{Z}]_*$ is the kernel of d, i.e. $\widetilde{K}(X)$.

One way to see that $\pi_1(BU) = 0$ is to use the fundamental equivalence $G \simeq \Omega BG$ valid for any topological group G, plus the fact that U is connected. You can deduce that from the fact that $\pi_0(U(i)) = \pi_0(U)$ for all $i \geq 1$ (as follows from the fiber sequences $U(n-1) \to U(n) \to S^{2n-1}$, see [**Swi75**, 11.36]) and $\pi_0(U(1)) = \pi_0(S^1) = 0$.

Remark 3.57. You may be wondering what is $[X, BU]$ if X is not connected. It can be identified with the abelian subgroup of $K(X)$ consisting of virtual vector bundles whose virtual dimension is zero over every point, not just over the basepoint. More precisely, there is a function $\mathrm{Vect}(X) \to [X, \mathbb{N}]$ which takes $[\xi]$ to the function that takes a point to the dimension of the fiber of ξ over it. It's a morphism of monoids so it induces a homomorphism $K(X) \to [X, \mathbb{Z}]$. Its kernel is the abelian group denoted $\widehat{K}(X)$, and is isomorphic to $[X, BU]$. It coincides with $\widetilde{K}(X)$ when X is connected.

Moreover, while we have $\widetilde{K}(X) \cong \mathcal{E}(X)$, the group $\widehat{K}(X)$ satisfies something similar. Let $\mathrm{Vect}^k(X)$ denote the monoid of isomorphism classes of complex k-plane vector bundles. Let $\mathrm{Vect}^k(X) \to \mathrm{Vect}^{k+1}(X)$ be the map that takes $[\xi]$ to $[\xi \oplus \epsilon^1]$. Let $\mathrm{Vect}^{(s)}(X) = \mathrm{colim}_k \mathrm{Vect}^k(X)$. It is a commutative monoid with the Whitney sum. One can prove that $\mathrm{Vect}^{(s)}(X) \cong \widehat{K}(X)$, so in particular $\mathrm{Vect}^{(s)}(X)$ is an abelian group.

See [**AGP02**, 9.4] for details.

Remark 3.58. The group $[X, BU \times \mathbb{Z}]_*$ exists for all X, not necessarily compact. That's one extension of the classical $\widetilde{K}(X)$ to non-compact spaces X. It's sometimes called *representable (topological) K-theory*, so as to distinguish it from other extensions.

Remark 3.59. Summarizing: Let X be a compact space. Consider a map $f : X \to BU$. If X is connected and pointed, then by Proposition 3.53 we know it defines an element in $\widetilde{K}(X) \cong \mathcal{E}(X)$, so, a stable class of vector bundles over X. It makes sense to call f the (classifying map of) a stable complex vector bundle. We typically drop the parentheses.

If X is not connected and pointed, the terminology still makes sense by Remark 3.57. Indeed, we then have that $[X, BU] \cong \widehat{K}(X) \cong \mathrm{Vect}^{(s)}(X)$, and it is also fair to call an element of $\mathrm{Vect}^{(s)}(X)$ a stable complex vector bundle.

Alternatively, by the same remark, a map $X \to BU$ classifies a virtual vector bundle of virtual dimension zero over every point.

In light of (3.56), if $BU \times \mathbb{Z}$ were the 0-space of an Ω-spectrum E (or, in other words, an infinite loop space), then the above says that $\widetilde{K}(X)$ is $E^0(X)$, at least when X is compact.

Let us now see that this indeed holds.

3.4. As a spectrum. The following is a fundamental theorem. As such, it has many different but equivalent formulations, and many different proofs, see MO:Proofs of Bott periodicity. A couple of textbook proofs: [**AGP02**, 9.5.1] and [**MP12**, 21.6.1].

Theorem 3.60 (Bott periodicity)**.** *There is a homotopy equivalence $BU \times \mathbb{Z} \simeq \Omega^2 BU$.*

Corollary 3.61. *There exists an Ω-spectrum KU, called the* periodic complex topological K-theory spectrum, *with spaces*

$$KU_n = \begin{cases} BU \times \mathbb{Z} & \text{if } n \text{ is even,} \\ U & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. We need to specify two (weak) homotopy equivalences: $BU \times \mathbb{Z} \xrightarrow{\sim} \Omega U$ and $U \xrightarrow{\sim} \Omega(BU \times \mathbb{Z})$.

For the first one: $BU \times \mathbb{Z} \simeq \Omega^2 BU = \Omega\Omega BU \simeq \Omega U$.

For the second one: note that loop spaces ΩX only depend on the connected component of the basepoint X_0, i.e. $\Omega X \cong \Omega X_0$. So $U \simeq \Omega BU \cong \Omega(BU \times \mathbb{Z})$. □

Therefore, KU defines a cohomology theory $KU^n(-) : \mathrm{Ho}(CW_*) \to \mathrm{Ab}$, $n \in \mathbb{Z}$, called *topological K-theory*, such that $KU^0(X) \cong \widetilde{K}(X)$ when X is a finite CW complex. Moreover, $KU^{2n}(X) = KU^0(X)$, and $KU^{2n+1} = KU^1(X) = KU^0(\Sigma X)$. There's therefore only 2 groups, really, and they repeat with periodicity 2, hence the periodicity in "Bott periodicity".

Corollary 3.62. $\pi_k(BU) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \text{ is odd,} \\ \mathbb{Z} & \text{if } k > 0 \text{ is even} \end{cases}$ *and so* $\pi_k(BU \times \mathbb{Z}) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \mathbb{Z} & \text{if } k \text{ is even.} \end{cases}$

PROOF. We know that $\pi_0(BU) = 0$, and for $k \geq 1$,

$$\pi_k(BU) \cong \pi_k(BU \times \mathbb{Z}) \cong \pi_k(\Omega^2 BU) \cong \pi_{k+2}(BU),$$

so the result follows from $\pi_2(BU) \cong \pi_0(BU \times \mathbb{Z}) \cong \mathbb{Z}$ and $\pi_1(BU) = 0$ which we have seen above. □

Remark 3.63. The real counterpart to KU is also important. It is denoted KO, and it also follows from a periodicity theorem of Bott. It states that $BO \times \mathbb{Z} \simeq \Omega^8 BO$: in this case, it's 8-periodicity. The homotopy groups of KO from 0 to 7 are $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$.

Seeing a complex vector space as a real one of double the dimension ("realification") gives maps $U(n) \to O(2n)$, which begets a map $U \to O$ and hence $BU \times \mathbb{Z} \to BO \times \mathbb{Z}$, and finally a map $KU \to KO$ and $KU^0(X) \to KO^0(X)$. Going the other way, there is a "complexification" map $c : KO \to KU$ which is a Galois extension in a appropriate sense [**Rog08**, 5.3.1].

Remark 3.64. Let us organize the thoughts above a bit differently. Consider the bijection $\mathrm{Vect}^n(X) \cong [X, BU(n)]$. Since $BU(n)$ is a connected H-space, then this is also $[X, BU(n)]_*$ [**Hat02**, 4A.2]. Applying this to S^k, we see $\mathrm{Vect}^n(S^k) \cong \pi_k(BU(n))$. So, the computation of the geometric $\mathrm{Vect}^n(S^k)$ is equated with a computation in homotopy theory, which is doable. For example, $\mathrm{Vect}^k(S^1) \cong \pi_1(BU(k)) = 0$ for $k \geq 1$, so we deduce that every complex vector bundle on S^1 is trivial.

We could play the same game over the reals, getting $\mathrm{Vect}^n_{\mathbb{R}}(S^1) \cong \pi_1(BO(k)) \cong \mathbb{Z}/2$ for all $n \geq 1$. For example, there are two isomorphism classes of real line bundles over S^1, the trivial one and the Möbius one.

3.5. The Hopf invariant 1 problem. Topological K-theory was used in the 50s and 60s to solve some outstanding problems in algebraic topology, like the Hopf invariant 1 problem, by Adams. We sketch what this is about, and some of its consequences. An exposition of all this can be found in [**Hat17**, 2.3].

Let $n > 1$ and $m > n$. Take a map $f : S^m \to S^n$, and take its homotopy cofiber, i.e. attach an $(m+1)$-cell to S^n via f:

$$\begin{array}{ccc} S^m & \xrightarrow{f} & S^n \\ \downarrow & & \downarrow \\ D^{m+1} & \longrightarrow & Cf \end{array}$$

Now, Cf has three cells: one in dimension 0, one in dimension n, and one in dimension $m+1$. Using cellular cohomology, we see that

$$\widetilde{H}^k(Cf) \cong \begin{cases} \mathbb{Z} & \text{if } k = n, m+1 \\ 0 & \text{else.} \end{cases}$$

What about the cup product? Let α be a generator of $\widetilde{H}^n(Cf)$ and β be a generator of $\widetilde{H}^{m+1}(Cf)$. By dimension reasons, the only cup product that has the chance of being non-zero is $\alpha \cup \alpha$, and only whenever $m+1 = 2n$. The number n should also be even, since if it's odd, then the graded-commutativity of the cup product implies that $\alpha^2 = -\alpha^2$, i.e. $\alpha = 0$.

Let's suppose $m+1 = 2n$, so $f : S^{2n-1} \to S^n$. Define $H(f) \in \mathbb{Z}$ to be such that

$$\alpha \cup \alpha = H(f)\beta.$$

Now, the sign of this number depends on how we choose β, but we can specify β in a way that doesn't make any choices, so $H(f)$ is well-defined.

If $f \simeq g$, then $H(f) = H(g)$, so we get a function $H : [S^{2n-1}, S^n] \to \mathrm{Ab}$. Since S^n is simply connected, then $[S^{2n-1}, S^n] \cong \pi_{2n-1}(S^n)$[8]. And in fact, one can prove [**Hat02**, 4B.1] that

$$H : \pi_{2n-1}(S^n) \to \mathrm{Ab}$$

is a homomorphism.

This is a homotopy invariant, so it may be useful for determining whether two maps $S^{2n-1} \to S^n$ are not homotopical: it suffices to compute their Hopf invariant and see that it's not the same. Computing homotopy groups is hard, so any tool is welcome. Now, here's a question, the famous "Hopf invariant one" question:

Does there exists an f with $H(f) = 1$?

Theorem 3.65 (Adams, 1960)**.** *There exists an f with $H(f) = 1$ if and only if $n = 2, 4, 8$.*

PROOF. Adams' original proof was long and used the complicated theory of secondary cohomological operations, but later Adams and Atiyah (1966) gave a new proof, much shorter, that uses topological K-theory (and its primary cohomological operations), see [**Hat17**, 2.3] or [**AGP02**, 10.6]. It uses a different definition of the Hopf invariant, actually, that directly appeals to K-theory instead of to integral cohomology. □

The maps with Hopf invariant one are the following. First, there is the Hopf map $f : S^3 \to S^2 = \mathbb{C}P^1$. Its mapping cone is the complex projective space $Cf = \mathbb{C}P^2$. We can compute its

[8]This is not the definition of π_n, since recall that in this section $[X, Y]$ denotes unpointed homotopy classes. This is an exercise, but see [**Hat02**, 4A.2] for an interesting general statement.

cohomology ring, it's $H^*(\mathbb{C}P^2) = \mathbb{Z}[\alpha]/\alpha^3$, $|\alpha| = 2$: this is a fundamental computation, see [**Hat02**, 3.19] or [**Swi75**, 15.33]. [9]

The other two maps are analogous, only using the quaternions $\mathbb{H}$, i.e. a map $S^7 \to S^4 = \mathbb{H}P^1$, and the octonions $\mathbb{O}$, i.e. a map $S^{15} \to S^8 = \mathbb{O}P^1$. You probably know the quaternions from your algebra class, they form a non-commutative algebra over $\mathbb{R}$ of dimension 4. The octonions are a bit more exotic because they are not even associative; they have dimension 8. The constructions of the three maps are in [**Hat02**, 4.45, 4.46, 4.47], they are all known as "Hopf fibrations".

Here are some interesting consequences, proofs of which you can find in [**Hat17**], see also [**AGP02**, 10.6] for more information on the relation between these statements and their history.

(1) $\mathbb{R}^n$ is an $\mathbb{R}$-algebra with division only when $n = 1, 2, 4, 8$. This purely algebraic question had been asked around the year 1900.
(2) S^n is an H-space only when $n = 0, 1, 3, 7$.
(3) S^n is parallelizable only when $n = 0, 1, 3, 7$, i.e. the tangent bundle to S^n is trivial only in those cases, i.e. S^n admits n vector fields linearly independent in each point only in those cases.
(4) The only fiber bundles where base space, total space and fiber are spheres are the three ones above, plus the real one $S^0 \to S^1 \xrightarrow{a} S^1 = \mathbb{R}P^1$ where a is the antipodal map, seeing $S^1 \subseteq \mathbb{R}^2$.

The fairly easy relations between all these statements can be found in [**AGP02**, 10.5].

Remark 3.66. When n is even, for any even number d there exists an f with $H(f) = d$. See [**Hat02**, Page 428]. As a corollary, since H is a homomorphism of groups, then an f with $H(f) = 2$ generates an infinite cyclic subgroup, so $\pi_{2n-1}(S^n)$ contains a copy of $\mathbb{Z}$ as a summand.

[9] As a commentary, the K-theory is the same [**Hat17**, 2.24]... both KU and $H\mathbb{Z}$ are *complex-oriented cohomology theories*, see MO:Motivation for complex orientable, where motivation and bibliography pointers are given. This is the starting point for a very deep theory.

CHAPTER 4

The homotopy theory of spectra

1. Localizations of categories with weak equivalences

I explained before that there are diferent notions of spectra, all equivalent in a homotopy-theoretical sense. I made a choice when I settled for a particular version. We reach here a similar point in the theory. Just as "what is a spectrum?" admits several answers, the question "what is a homotopy theory?" also does. The simplest answer is: a category $\mathcal{C}$ together with a class of morphisms $\mathcal{W}$ called *weak equivalences* (which "somewhat resemble isomorphisms but fail to be invertible in any reasonable sense, and might in fact not even be reversible: that is the presence of a weak equivalence $X \xrightarrow{\sim} Y$ need not imply the presence of a weak equivalence $Y \xrightarrow{\sim} X$", to copy the words of Riehl [**Rie19**]).[1] There is a formal procedure to invert those arrows, getting a category $\mathcal{C}[\mathcal{W}^{-1}]$, the *localization of* $\mathcal{C}$ *at* $\mathcal{W}$ (think of the localization of a ring at a multiplicative subset). You might want to think of that as the "homotopy category" of $\mathcal{C}$, though the terminology is a bit abusive as there is no notion of homotopy between maps here.

Proposition 4.1. *Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a class of morphisms in $\mathcal{C}$ called "weak equivalences". There exists a (not necessarily locally-small[2]) category $\mathcal{C}[\mathcal{W}^{-1}]$ and a functor $\iota : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ such that: ι sends the maps in $\mathcal{W}$ to isomorphisms; if $\mathcal{C} \to \mathcal{D}$ is a functor that sends $\mathcal{W}$ to isomorphisms, then there exists a unique functor making the following diagram commute*

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ {\scriptstyle\iota}\big\downarrow & \nearrow & \\ \mathcal{C}[\mathcal{W}^{-1}]. & & \end{array}$$

This implies that $\mathcal{C}[\mathcal{W}^{-1}]$ is the unique category with a functor from $\mathcal{C}$ satisfying the above two properties, up to a unique isomorphism of categories making the obvious triangle commute.

Isomorphisms of categories can be characterized as follows: $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism of categories if and only if it is fully faithful and bijective on objects.

Side remark 4.2 (For category theory amateurs)**.** You may be worrying about two things in the previous proposition. One is: this universal property says only that

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \xrightarrow{\iota^*} \mathrm{Fun}^{\sim \mapsto \cong}(\mathcal{C}, \mathcal{D})$$

is an isomorphism of sets, where the right-hand side means functors which send weak equivalences to isomorphisms... But these are categories; you may want ι^* to be an isomorphism of categories. This is automatic MO:74955. Moreover, you may worry that this is too strict, and that we should require that ι^* just be an *equivalence* of categories, which would give e.g. that $\mathcal{C}[\mathcal{W}^{-1}]$ is unique only up to equivalence, and which would say that the dotted functor is unique only up to natural isomorphism. See MO:312118.

[1] Some authors require more axioms, like $\mathcal{W}$ being a subcategory containing all objects, or the weak equivalences satisfying the 2-out-of-3 property.

[2] This means that the "hom-sets" may actually be "hom-classes".

Think of the construction of localization of rings, which satisfies an analogous universal property. If the ring R is commutative and S is a multiplicative subset, we can define $S^{-1}R$ to consist of symbols rs^{-1}, $s \in S, r \in R$, subject to some relations. If R is not commutative, it's not that simple, because $r_0 s_0^{-1} r_1 s_1^{-1}$ is not necessarily of the form rs^{-1}... so if we define $S^{-1}R$ to be the sums of sequences of the form $r_0 s_0^{-1} r_1 s_1^{-1} r_2 s_2^{-1} \cdots$. In our case, we will need to do something similar.

PROOF. We will give an explicit description of $\mathcal{C}[\mathcal{W}^{-1}]$.

The objects of $\mathcal{C}[\mathcal{W}^{-1}]$ are the same as those of $\mathcal{C}$. For the arrows, we take finite strings of arrows of $\mathcal{C}$ or of formal inverses of arrows of $\mathcal{W}$: these are "zig-zags" that we can represent like this

$$X \xleftarrow{\sim} X_1 \longrightarrow X_2 \xleftarrow{\sim} X_3 \xleftarrow{\sim} \cdots \longrightarrow X_n \longrightarrow Y$$

where $\sim$ signals that an arrow is in $\mathcal{W}$. Any given object of $\mathcal{C}$ may appear in the zig-zag, and the objects of $\mathcal{C}$ may form a proper class (not a set), which explains why $\mathcal{C}[\mathcal{W}^{-1}]$ may not be localy small. We impose on these zig-zags the equivalence relation generated by the following:

(1) Compose two adjacent maps pointing to the right,
(2) Remove any identity map pointing to the right,
(3) Remove any $Y \underset{f}{\xleftarrow{\sim}} X \underset{f}{\xrightarrow{\sim}} Y$ or $X \underset{f}{\xrightarrow{\sim}} Y \underset{f}{\xleftarrow{\sim}} X$.

Composition of (equivalence classes of) zig-zags is given by concatenation. The identity is the empty zig-zag. The functor $\iota : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ maps $X \to Y$ to $X \to Y$. The functor ι sends weak equivalences to isomorphisms: if $f : X \xrightarrow{\sim} Y$, its inverse in $\mathcal{C}[\mathcal{W}^{-1}]$ is $Y \xleftarrow{f} X$. Rewriting it as $Y \xrightarrow{\text{id}} Y \xleftarrow{f} X$, we can think of it as $1/f$; more generally, we can think of $X \xrightarrow{g} Y \xleftarrow{f} Z$ as the quotient g/f.

If $F : \mathcal{C} \to \mathcal{D}$ is another functor that inverts the weak equivalences, then define $\widetilde{F} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ as follows. Since any zig-zag is a composition of unary zig-zags, it suffices to define $\widetilde{F}$ on single arrows pointing to the right or to the left. Define $\widetilde{F}(X \to Y)$ as $F(X \to Y)$, and $\widetilde{F}(Y \xleftarrow{\sim} X)$ as $F(X \to Y)^{-1}$; this respects the relations so does define a functor $\widetilde{F}$.

If $G : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is another functor that extends F, then it has to coincide with $\widetilde{F}$. Indeed, they coincide on objects and on right-pointing arrows. As for left-pointing arrows, this follows immediately from property (3) above. ☐

Proposition 4.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with classes of weak equivalences $\mathcal{W}$ and $\mathcal{W}'$ respectively. If $F : \mathcal{C} \to \mathcal{D}$ preserves weak equivalences, then there is a unique induced functor $\widetilde{F} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}[\mathcal{W}'^{-1}]$, the* (total) derived functor *of F, that makes the following diagram commute.*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \iota \downarrow & & \downarrow \iota \\ \mathcal{C}[\mathcal{W}^{-1}] & \underset{\widetilde{F}}{\longrightarrow} & \mathcal{D}[\mathcal{W}'^{-1}]. \end{array}$$

If F has a right adjoint G that also preserves weak equivalences, then $(\widetilde{F}, \widetilde{G})$ is an adjoint pair. If moreover the unit $X \to GFX$ and the counit $FGY \to Y$ of (F, G) are weak equivalences, then $(\widetilde{F}, \widetilde{G})$ is an equivalence.

PROOF. Define $\widetilde{F}$ by applying F to every term of the zig-zag pointing to the right, and by taking $F(-)^{-1}$ on every term pointing to the left. To see that this is the unique functor fitting in the above commutative diagram, first note that by this commutativity it has to send a simple zig-zag $f : X \to Y$ to $Ff : FX \to FY$. Considering the zig-zag $X \xrightarrow{\text{id}} X \xleftarrow{\sim} Y$, we see that zig-zags $X \xleftarrow{\sim} Y$ have to be sent to $FX \xleftarrow{\sim} FY$. By further composing these, we get the result.

The additional adjunction properties are similarly easily proved. □

Example 4.4. In this example, $\mathrm{Ho}(\mathcal{C})$ means the category with the same objects as $\mathcal{C}$, and with maps given by homotopy classes of maps (in the examples below, we have a notion of homotopy).

(1) Let $\mathcal{C} = \mathrm{Top}$, $\mathcal{W} =$ the class of homotopy equivalences. Then it's not hard to prove that $\mathrm{Top}[\mathcal{W}^{-1}] \cong \mathrm{Ho}(\mathrm{Top})$ (isomorphism of categories). The key thing to observe, and we isolate it in a remark because it will be useful in the future, is that:

Remark 4.5. A functor $F : \mathrm{Top} \to \mathcal{D}$ takes homotopy equivalences to isomorphisms if and only if it takes homotopical maps to equal maps, as follows from considering the following diagram: $X \underset{i_1}{\overset{i_0}{\rightrightarrows}} \overset{\sim}{\leftarrow} X \times I \xrightarrow{H} Y$.

(2) $\mathcal{C} = \mathrm{Top}$, $\mathcal{W} =$ the class of weak homotopy equivalences. If X is not of the homotopy type of a CW-complex and you take a CW-approximation $QX \xrightarrow{\sim} X$, then $[X,Y] \ncong [QX,Y]$. In this example, it's the latter that appears as a hom.

Indeed, $\mathrm{Top}[\mathcal{W}^{-1}] \simeq \mathrm{Ho}(CW)$ (equivalence of categories), essentially by CW-approximation and Whitehead's theorem. As a technical point, note that here it's an equivalence of categories but in the previous example it was an isomorphism of categories. If you want a similar isomorphism as above in this example, you should take $\mathcal{H}$ to be the category with all spaces as objects, and arrows from X to Y to be $[QX, QY]$. Then $\mathrm{Top}[\mathcal{W}^{-1}] \cong \mathcal{H}$. [3]

(3) $\mathcal{C} = \mathrm{Ch}_R^{\geq 0}$ be the category of (homologically graded) non-negative chain complexes over a ring R. We can take homotopy equivalences as $\mathcal{W}$, in which case $\mathrm{Ch}_R^{\geq 0}[\mathcal{W}^{-1}] \cong \mathrm{Ho}(\mathrm{Ch}_R^{\geq 0})$ is sometimes denoted $\mathcal{K}(R)$ and called "the homotopy category of R" in homological algebra circles.

(4) $\mathcal{C} = \mathrm{Ch}_R^{\geq 0}$ again, and $\mathcal{W}$ is the class of quasi-isomorphisms. Then $\mathrm{Ch}_R^{\geq 0}[\mathcal{W}^{-1}] \simeq \mathrm{Ho}(\mathrm{Ch}_R^{\mathrm{proj}, \geq 0}) = \mathcal{D}(R)$, the derived category of R, where proj denotes complexes of projective modules.

Remark 4.6. (1) In the previous chapter, we used $\mathrm{Ho}(\mathrm{Top})$ to mean the usage in 1 above. But many people prefer to use it to mean what we have in 2 above. Perhaps the most fundamental reason to prefer that one is that it's equivalent to the standard homotopy category of simplicial sets. The usefulness of this is hard to overstress...

(2) Examples 1 and 3 above are very similar to each other, and examples 2 and 4 as well. For an exposition of the four model structures associated to these examples that highlights their similarities, see [**MP12**, Chapters 17 and 18].

Exercise 4.7. *In the context of the proposition above, prove that if F has a right adjoint G which also preserves weak equivalences, then $(\tilde{F}, \tilde{G})$ is an adjunction. Moreover, if the unit and the counit of (F, G) are weak equivalences, then $(\tilde{F}, \tilde{G})$ is an adjoint equivalence.*

2. Different approaches to homotopy theory

The approach to homotopy theory of Section 1 is sufficient, see [**BK12**]; it is also the most economical. It has several disadvantages if one wants to work with it, though. One of them is that the hom-classes in $\mathcal{C}[\mathcal{W}^{-1}]$ are a bit unwieldy and we should be careful not to run into set-theoretical paradoxes. This is inconvenient.

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[3] In practice we shall never take advantage of the fact that you can build such "homotopy categories" which are *isomorphic* to the localization: equivalence is what we really care about. But if you're interested, you should know that what I just said is general and works in any model category $\mathcal{C}$: see [**Rie19**, 3.4.4, 3.4.5].

Moreover, while any morphism in $\mathcal{C}$ trivially gives a morphism in $\mathcal{C}[\mathcal{W}^{-1}]$, there are some morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$ that may be hard to describe. You may be able to get an explicit zig-zag that describes it, all right. But there are some morphisms that are bound to be hard to describe like that: zig-zags may be very long.

In the case of localization of noncommutative rings, there are conditions such as the *Ore condition* which guarantee that elements in $S^{-1}R$ can be represented as rs^{-1} instead of as infinite chains. Similarly, those of you familiar with derived categories may know of the *calculus of fractions* of Gabriel–Zisman, which is analogous to that. If $\mathcal{W} \subseteq \mathcal{C}$ admits such a calculus, then the zig-zags in the homsets of the homotopy category can be drastically shortened: one arrow to the right followed by one to the left, where the arrows going to the left are weak equivalences.

Around the same time the *model categories*[4] of Quillen were introduced [**DS95**], [**Hov99**], [**MP12**]. These introduce extra structure: additionally to the weak equivalences, it is necessary to specify a class of cofibrations and a class of fibrations, subject to a bunch of axioms. Proving that something is a model category takes a bit of work, but then you get for free some very useful results which are valid in any model category. Here, the zigzags in $\mathrm{Hom}_{\mathcal{C}[\mathcal{W}^{-1}]}(X, Y)$ can be greatly shortened: we only need

$$X \xleftarrow{\sim} QX \longrightarrow RY \xleftarrow{\sim} Y.$$

(This is not a calculus of fractions but a *3-arrow calculus*, see [**DHKS04**, I.3.2], [**BK13**] for details.)

Moreover: there is a notion of homotopy between maps, and, above, $\mathrm{Hom}_{\mathcal{C}[\mathcal{W}^{-1}]}(X, Y)$ can also be presented as the set of maps $QX \to RY$ up to homotopy. Here QX is "cofibrant" and the weak equivalence $QX \to X$ is a "cofibrant approximation"; dually, RY is "fibrant" and $Y \to RY$ is a "fibrant approximation".

This recovers e.g. the traditional homotopy category of spaces (where we can omit both Q and R: a most rare occurrence in model category theory). The approach of Section 1 is rather formal, categorical, whereas this one starts to feel more like homotopy theory.

Model categories present additional advantages: for example, it is reasonably simple to produce a theory of derived functors, which generalizes their homological algebra counterpart. The problem here is that there are many interesting functors between categories with weak equivalences which do not preserve them, so it's not obvious how to use Proposition 4.3 to build the induced functor on homotopy categories. Think of the tensor product with a fixed module, between categories of chain complexes.

Model categories also have disadvantages. For example, categories of functors to a model category cannot be endowed with sensible model structures in full generality, which complicates a bit the development of the theory of homotopy (co)limits. One fix for that is given by the *homotopical categories* of [**DHKS04**], see also [**Shu06**]. These are categories with weak equivalences satisfying a couple of mild axioms, which prove to be enough for a surprising amount of things... but which still have the problem of hom-sets consisting of zig-zags. It's better to be eclectic.

Side remark 4.8. There is another family of approaches to the notion of "homotopy theory", which does not rely on specifying a class of weak equivalences. It grows from a different observation: in the category of spaces, you have maps of spaces, but also maps between maps of spaces (homotopies), and homotopies between homotopies, and so on, and all of these homotopies are invertible. Spaces are thus the prototype of an $(\infty, 1)$-category, similarly to how

[4]Full name: category of models for a homotopy theory.

Set is the prototypical 1-category, and an $(\infty, 1)$-category can be thought of as "a homotopy theory".

Again, there are many incarnations of what is an $(\infty, 1)$-category: categories enriched in simplicial sets, or in topological spaces, are pretty strict (or "algebraic") models, whereas other models are more flexible (or "geometric"), e.g. complete Segal spaces or quasicategories, called simply ∞-categories by Lurie. See e.g. the surveys [**AC16**] or [**Rie19**].

We will not delve further on these important issues of abstract homotopy theory, because in this course we are focusing on one homotopy theory: that of spectra. Therefore, we have chosen not to develop any of the above approaches to abstract homotopy theory. The model-categorical one, though, will be lingering in the background. We shall make passing allusions to it, which you can ignore if you don't find them illuminating.

3. First definition of the stable homotopy category

Consider examples 2 and 4 of Example 4.4. In both of those localized categories, the hom-sets are given by $[QX, Y]$, where Q either means CW-approximation or projective resolution. In other words, to get maps from X, if X is not already a CW-complex or a projective chain complex, I need to approximate it by one, and then take homotopy classes of maps out of it.

For spectra, we will follow a similar path. We have defined maps of spectra $X \to Y$. We will now define weak equivalences – so the homotopy category of spectra can be defined as $\mathrm{Sp}[\mathcal{W}^{-1}]$, i.e. the maps in it are zig-zags. We will define cofibrant spectra (analogous to "CW-complex" in spaces or to "projective" in chain complexes) and fibrant spectra (in the spaces and chain complexes examples above, every object is fibrant, so this complication is non-existent); every spectrum will have a weak equivalence from/to a co/fibrant spectrum; and $\mathrm{Hom}_{\mathrm{Sp}[\mathcal{W}^{-1}]}(X, Y)$ will be bijective to the homotopy classes of maps between these approximations.

We now define the "weak equivalences" of spectra. These are defined similarly as weak homotopy equivalences of spaces or quasi-isomorphisms of chain complexes[5]. So we first need to define homotopy groups of spectra: note that these are defined for any integer, not only for the non-negative ones!

Definition 4.9. Let $k \in \mathbb{Z}$ and X be a spectrum. Define

$$\pi_k(X) = \mathrm{colim}_i \pi_{k+i}(X_i).$$

More explicitly, one step in this sequence is given by the map

$$\pi_{k+i}(X_i) \to \pi_{k+i+1}(\Sigma X_i) \xrightarrow{(\rho_i)_*} \pi_{k+i+1}(X_{i+1})$$

where the first map is obtained by suspension and the second one from the structure map. Equivalently, if η is the unit of the (Σ, Ω) adjunction in pointed spaces, then the first map is given by $\pi_{k+i}(X_i) \xrightarrow{\eta_*} \pi_{k+i}(\Omega\Sigma X_i) \cong \pi_{k+i+1}(\Sigma X_i)$. Yet another description: it is the composition $\pi_{k+i}(X_i) \xrightarrow{(\tilde{\rho}_i)_*} \pi_{k+i}(\Omega X_{i+1}) \cong \pi_{k+i+1}(X_{i+i})$.

This definition readily extends to a functor $\pi_k : \mathrm{Sp} \to \mathrm{Ab}$.

Go to Table 1 to see all of these homotopy groups and maps between them in a table.

[5] They're actually closer to the later, in the sense that, as we will see in a moment, they are both of the form "the class of maps that gets inverted by a certain functor from your category". It would be great to say the same thing is true for homotopy groups of spaces, but you have to take all basepoints when your spaces are not connected!

TABLE 1. Homotopy groups of a spectrum

	X_0	X_1	X_2	X_3	
π_0	$\pi_0(X_0)$	$\pi_0(X_1)$	$\pi_0(X_2)$	$\pi_0(X_3)$	
π_1	$\pi_1(X_0)$	$\pi_1(X_1)$	$\pi_1(X_2)$	$\pi_1(X_3)$	$\pi_{-3}(X)$
π_2	$\pi_2(X_0)$	$\pi_2(X_1)$	$\pi_2(X_2)$	$\pi_2(X_3)$	$\pi_{-2}(X)$
π_3	$\pi_3(X_0)$	$\pi_3(X_1)$	$\pi_3(X_2)$	$\pi_3(X_3)$	$\pi_{-1}(X)$
		$\pi_3(X)$	$\pi_2(X)$	$\pi_1(X)$	$\pi_0 X$

Definition 4.10. We say that a spectrum X is *connective*[6] if all its negative homotopy groups vanish.

Example 4.11. Let $k \in \mathbb{Z}$ and X a pointed space. Then

$$\pi_k(\Sigma^\infty X) = \begin{cases} \pi_k^s(X) & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

by definition in the $k \geq 0$ case, and by the higher connectivity properties of iterated suspensions in the $k < 0$ case. Therefore, suspension spectra are connective.

In particular, the sphere spectrum $\mathbb{S}$ is connective and its positive homotopy groups $\pi_k(\mathbb{S}) = \pi_k^s(S^0)$ are the stable stems, discussed in Section 2.1.

Exercise 4.12. *Prove that* $\pi_k(X \times Y) \cong \pi_k(X) \times \pi_k(Y)$.

Remark 4.13. If X is an Ω-spectrum, then in the sequence defining its homotopy groups, all the maps are isomorphisms: if $k \geq 0$, then,

$$\pi_k(X) = \operatorname{colim}(\ \pi_k(X_0) \xrightarrow{\cong} \pi_{k+1}(X_1) \xrightarrow{\cong} \pi_{k+2}(X_2) \xrightarrow{\cong} \cdots\ )$$

whereas for $-k \leq 0$, we need to start the sequence later than at X_0 (because negative homotopy groups of spaces don't exist):

$$\pi_{-k}(X) = \operatorname{colim}(\ \pi_0(X_k) \xrightarrow{\cong} \pi_1(X_{k+1}) \xrightarrow{\cong} \pi_2(X_{k+2}) \xrightarrow{\cong} \cdots\ ).$$

In conclusion, the canonical map $\pi_{-k+n}(X_n) \xrightarrow{\cong} \pi_{-k}(X)$ is an isomorphism for all $n \geq 0$ such that $-k+n \geq 0$, which is natural in X. In Table 1, all the groups of the same color are isomorphic.

Example 4.14. The homotopy groups of the Eilenberg–Mac Lane spectrum HG of an abelian group G are given by:

$$\pi_n(HG) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

Side remark 4.15. Loop spaces are examples of grouplike A_∞-spaces (Side remark 2.10), also known as E_1-groups. Double loop spaces are examples of E_2-groups: that is, E_1 plus homotopy-commutativity. Higher loop spaces are examples of E_n-groups: they are homotopy-commutative in a structured way, up to degree n.

Think of monoidal categories: you have plain monoidal categories, braided monoidal categories, then symmetric monoidal categories: after that, it stabilizes, there is nothing "more commutative" than symmetric monoidality.

Here, there is no stabilization, only an ever-greater degree of commutativity... up to infinity. Infinite loop spaces, as defined above, are examples of E_∞-groups. In the previous remark, we

[6]This is not a typo. We do not mean "connected". If we were to use the terminology of connectedness, then "connective" corresponds to (-1)-connected, which is a bit of a moutfhul, hence the alternative term for it.

saw that X is pretty close to X_0, at least if X is connective. Indeed, the homotopy groups of the former are the stable homotopy groups of the latter. Much more is true: once we look at X_0 not only as a space but as an E_∞-group, then it is actually capturing all the homotopical information of the connective spectrum X. A bit more precisely, taking the 0-space of a connective spectrum establishes an equivalence (in a homotopical sense) between them and E_∞-groups. This is a very important recognition theorem, but it takes a fair amount of technology to even get a formal statement (see e.g. [**Lur17**, 5.2.6.26], or [**Ada78**, 2.3.2] for a more leisurely and classical treatment), so we shall not study it in this course. There are similar recognition theorems for E_n-groups as iterated loop spaces.

Definition 4.16. A map $f : X \to Y$ of spectra is an *equivalence* (or "weak equivalence", or "stable equivalence") if it induces an isomorphism in all homotopy groups.

If we denote by $\mathcal{W}$ the class of weak equivalences in Sp, then we have a category $\mathrm{Sp}[\mathcal{W}^{-1}]$ as per Proposition 4.1. This is the *homotopy category of spectra*, also known as *the stable homotopy category*.

We say that X is *equivalent* to Y and we write $X \simeq Y$ if X is isomorphic to Y in $\mathrm{Sp}[\mathcal{W}^{-1}]$, i.e. there exists a zig-zag of weak equivalences between X and Y.

Example 4.17. Let $p \in \mathbb{Z}$. Take a degree p map $f : S^1 \to S^1$, and consider its suspensions $\Sigma^n f : S^n \to S^n$ for all $n \geq 0$. Since the suspensions of f are also of degree p (check!), we would like to say that we have defined a map $\mathbb{S} \to \mathbb{S}$ of degree p. But we haven't defined it in level 0, and there is no way to extend what we have defined to degree 0. So with our definition of a morphism, we are not capturing this very elementary one. Similarly, we would like the Hopf map $S^3 \to S^2$ to induce a map $\Sigma\mathbb{S} \to \mathbb{S}$, but this map is not the suspension of a map $S^2 \to S^1$ (those are all nullhomotopic).

Here's one way to solve this problem:

Example 4.18. Let X and Y be spectra and $f : X \to Y$ be a map of spectra. If f_n is a weak homotopy equivalence for all large enough n, then f is a stable equivalence.

In particular, let $k \geq 0$ and define X' as having $*$ in levels $\leq k-1$, and having X_i in levels $i \geq k$, with structure maps coming from those of X. Then the obvious map $X' \to X$ is a stable equivalence (see remark Remark 4.19).

This shows that when defining a spectrum, only what happens "from a certain point on" is important, up to stable equivalence.[7]

This gives a first solution to the problem in Example 4.17: let $\mathbb{S}'$ be the truncation as above, where $k = 1$. We can define a map $p : \mathbb{S}' \to \mathbb{S}'$ "of degree p" as in that example. Then $\mathbb{S} \xleftarrow{\sim} \mathbb{S}' \xrightarrow{p} \mathbb{S}' \xrightarrow{\sim} \mathbb{S}$ is a "degree p" map $\mathbb{S} \to \mathbb{S}$ in $\mathrm{Sp}[\mathcal{W}^{-1}]$.

Remark 4.19. We just used the easy fact that if there exists an $N \geq 0$ such that $f_n : X_n \to Y_n$ is a weak equivalence for all $n \geq N$, then f is a weak equivalence.[8] A map $f : X \to Y$ of spectra is a *level equivalence*[9] if $f_n : X_n \to Y_n$ is a weak equivalence for all $n \geq 0$. A level equivalence is a stable equivalence, and the example above proves that the converse doesn't hold in general.

Proposition 4.20. *Let $f : X \to Y$ be a map between Ω-spectra. If f is a stable equivalence, then it is a level equivalence.*

PROOF. This follows immediately from Remark 4.13. □

[7] It also shows why our definition of a spectrum is equivalent to the definition of other authors that use X_n for negative n as well.

[8] We can generalize this to a cofinal subsequence of $\mathbb{N}$.

[9] This is not a genuinely interesting notion, but useful for some developments of the theory.

Remark 4.21. As we said in Section 2, the model-categorical approach to homotopy theory will be lurking in the background. The statement which we won't formally state nor prove is: there is a model category structure on Sp with weak equivalences given by the stable equivalences. The fibrations will be levelwise Serre fibrations, and the cofibrations will be retracts of relative cell spectra. The fibrant objects are the Ω-spectra, and the cofibrant objects are retracts of cell spectra: this will be explained below. This model structure was first established for spectra of simplicial sets in [**BF78**], see also [**GJ99**, X.4]. The topological version was made explicit in [**MMSS01**, 9.2], and is exposed in lush detail in nLab:model struture on topological sequential spectra.

4. Fibrant replacement

To understand why Ω-spectra will be useful in solving the problem from Example 4.17, suppose we have a spectrum X, an Ω-spectrum Y, and maps $f_k : X_k \to Y_k$ defined for $k \geq 1$ and which are compatible with the structure maps. Consider the zig-zag

$$\begin{array}{ccc} X_0 & & Y_0 \\ \downarrow & & \downarrow \sim \\ \Omega X_1 & \xrightarrow[\Omega f_1]{} & \Omega Y_1 \end{array}$$

If $Y_0 \to \Omega Y_1$ were an actual homeomorphism, then we would have an actual map $f_0 : X_0 \to Y_0$, and thus an actual map of spectra $X \to Y$. This is the approach taken by Peter May, see e.g. [**LMSM86**, Preamble]. Our Ω-spectra are weaker[10], so in order to fully solve this problem we need to work a bit more. We will come back to this in Example 4.26 and Remark 4.60.

Definition 4.22. Define a functor $R : \mathrm{Sp} \to \mathrm{Sp}$ with values in Ω-spectra as follows. Let $(RX)_n$ be the telescope (homotopy colimit; see Section 1.5) of the sequence

$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \cdots$$

taken over the loops of the transposes of the structure maps $\Sigma X_n \to X_{n+1}$. The transpose structure maps are:

$$\begin{aligned} (RX)_n &= \operatorname{hocolim}(X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \cdots) \\ &\simeq \operatorname{hocolim}(\Omega X_{n+1} \to \Omega^2 X_{n+2} \to \cdots) \\ &\xrightarrow{\sim} \Omega \operatorname{hocolim}(X_{n+1} \to X_{n+2} \to \cdots) \\ &= \Omega(RX)_{n+1} \end{aligned}$$

where the first $\simeq$ is a homotopy equivalence[11], and where the universal map from the telescope is a weak homotopy equivalence since homotopy groups take telescopes to colimits (see Section 1.5).

Remark 4.23. Compare to the definition of QY for a based space Y: the natural map $(R\Sigma^\infty Y)_n \to Q\Sigma^n Y$ is a weak homotopy equivalence; remark that $Q\Sigma^n Y$ is defined by the strict colimit instead of the homotopy colimit. In particular, we have a weak homotopy equivalence $(R\Sigma^\infty Y)_0 \to QX$.

[10] And as a result, our functor R below is much simpler than May et al's *spectrification functor* L which turns a spectrum into an equivalent Ω-spectrum with homeomorphisms as structure maps.

[11] Possibly killing a mosquito with a sledgehammer, this is an application of the homotopy cofinality theorem [**MV15**, 8.6.5], see also [**Dug08**, Section 6].

Going more general than suspension spectra, then unlike in the definition of QY, we really need to take the telescope in the construction of R if we want homotopy groups to commute, since the maps $X_n \to \Omega X_{n+1}$ are not closed inclusions, in general.

Following Remark 3.19, $\Omega^\infty(RX) = (RX)_0$ is an infinite loop space.

Now, RX is not just any Ω-spectrum: it is an Ω-spectrum *replacement* of X, in the sense that:

Proposition 4.24. *Let X be a spectrum. There is a natural equivalence $X \to RX$.*

PROOF. First, note that X_n is homotopy equivalent to the telescope of the constant sequence $X_n \xrightarrow{\text{id}} X_n \xrightarrow{\text{id}} \cdots$. Define a map $X_n \to (RX)_n$ by taking the telescope of the following commutative ladder:

$$\begin{array}{ccccccc}
X_n & \xrightarrow{\text{id}} & X_n & \xrightarrow{\text{id}} & X_n & \xrightarrow{\text{id}} & \cdots \\
\downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \widetilde{\rho}_n} & & \downarrow{\scriptstyle \Omega\widetilde{\rho}_{n+1}\circ\widetilde{\rho}_n} & & \\
X_n & \xrightarrow[\widetilde{\rho}_n]{} & \Omega X_{n+1} & \xrightarrow[\Omega\widetilde{\rho}_{n+1}]{} & \Omega^2 X_{n+2} & \longrightarrow & \cdots
\end{array}$$

where $\widetilde{\rho}_n$ denote the transpose structure maps. It assembles to a map of spectra $X \to RX$ (check!), which is a weak equivalence by a "colimits commute with colimits" argument:

$$\begin{aligned}
\pi_k(RX) &= \text{colim}_i \pi_{k+i}((RX)_i) \\
&\cong \text{colim}_i \text{colim}_j \pi_{k+i+j}(X_{i+j}) \\
&\cong \text{colim}_j \text{colim}_i \pi_{k+i+j}(X_{i+j}) \\
&\cong \text{colim}_j \pi_k(X) \cong \pi_k(X).
\end{aligned}$$

□

The functor R is our *fibrant replacement* functor, and Ω-spectra can equivalently be called *fibrant spectra*.

Corollary 4.25. *The functor R takes weak equivalences to weak equivalences.*

PROOF. Follows from the 2-out-of-3 property of weak equivalences, and naturality of $X \to RX$. □

Example 4.26. Let us see how this fibrant replacement helps us fix the problem from Example 4.17. Let's concentrate on the Hopf map for simplicity. We will define a map $\eta : R\Sigma\mathbb{S} \to R\mathbb{S}$ which can be thought of as the Hopf map at the level of sphere spectra.

Define the map $(R\Sigma\mathbb{S})_0 \to (R\mathbb{S})_0$ as the homotopy colimit of the following vertical maps:

$$\begin{array}{ccccccccc}
S^1 & \longrightarrow & \Omega S^2 & \longrightarrow & \Omega^2 S^3 & \longrightarrow & \Omega^3 S^4 & \longrightarrow & \cdots \\
 & & & & \downarrow{\scriptstyle \Omega^2 h} & & \downarrow{\scriptstyle \Omega^3\Sigma h} & & \\
S^0 & \longrightarrow & \Omega S^1 & \longrightarrow & \Omega^2 S^2 & \longrightarrow & \Omega^3 S^3 & \longrightarrow & \cdots
\end{array}$$

where $h : S^3 \to S^2$ is the Hopf fibration. Define $(R\Sigma\mathbb{S})_n \to (R\mathbb{S})_n$ similarly for $n \geq 1$. Note in $n = 1$ you can start the vertical maps from the second column, and from $n = 2$ you can already define them in all the columns. You can check it commutes with the structure maps, and so this defines the map η. From a categorical point of view, the key thing that has helped us solve the problem was the fact that to define a map between homotopy colimits, you may define them just from a certain point on.

Note this is a general procedure: if X and Y are spectra and $f_n : X_n \to Y_n$ for $n \geq k > 0$ is a compatible set of maps, then they can be used to define a map of spectra $RX \to RY$. We therefore have a zig-zag $X \xrightarrow{\sim} RX \to RY \xleftarrow{\sim} Y$.

How does this zig-zag compare to the one from Example 4.18? They are equivalent, in the sense that we can build a natural map of spectra $X' \to RX$ which is the inclusion of the basepoint $* \to (RX)_n$ in low degrees and is the map $X_n \to (RX)_n$ constructed above in high degrees. We get a commutative diagram

$$\begin{array}{ccccccc} X & \xleftarrow{\sim} & X' & \longrightarrow & Y' & \xrightarrow{\sim} & Y \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ X & \xrightarrow{\sim} & RX & \longrightarrow & RY & \xleftarrow{\sim} & Y \end{array}$$

where the vertical maps are weak equivalences, so the zig-zags are equivalent.

5. Cofibrant replacement

CW-complexes are very useful. The fact that they are defined in an inductive fashion is of practical help in many arguments; also, weak homotopy equivalences between CW-complexes are homotopy equivalences (Whitehead's theorem).

In spectra, we can make an analogous definition, and it plays an similar role.[12] First, a little caveat: we can be somewhat more general than CW-complexes and still have some of the same nice properties. We can do that in spaces, too. Here's the general definition:

Definition 4.27. Let $\mathcal{C}$ be a cocomplete category. Let $\mathcal{J}$ be a set of maps in $\mathcal{C}$, called *cells* and often omitted from the notation. A *$\mathcal{J}$-cell complex* is an object gotten as a colimit of a (possibly transfinite[13]) sequence of maps

$$\varnothing = X^{(-1)} \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \cdots$$

where $\varnothing$ denotes the initial object, and each of the maps $X^{(j)} \to X^{(j+1)}$ is a pushout

$$\begin{array}{ccc} \coprod_i S_i & \longrightarrow & \coprod_i D_i \\ \downarrow & & \downarrow \\ X^{(j-1)} & \longrightarrow & X^{(j)} \end{array}$$

where the $S_i \to D_i$ are in $\mathcal{J}$. In words: we build the $X^{(j)}$ inductively, starting with the initial object, and building $X^{(j)}$ from $X^{(j-1)}$ by gluing a bunch of cells $S \to D$ via *attaching maps* $S \to X^{(j-1)}$.

If $A \in \mathcal{C}$, a *relative $\mathcal{J}$-cell complex* $A \to X$ is defined as above, but replacing $X^{(-1)}$ by A. Note that A can be any object, it is not gotten by attaching cells.

Remark 4.28.
(1) A cell complex is a relative cell complex of the form $\varnothing \to X$.
(2) We could define a cell complex by attaching one cell at a time, instead of several, thus eliminating the coproducts from the definition; see [**MP12**, 15.1.3].[14]
(3) We can choose to stop at a given stage: reach only up to $X^{(n)}$ and then stop attaching.

Example 4.29. For the most trivial example, note that if $S \to D$ is a cell, then $S \to D$ is a relative cell complex, where we have attached only one cell to S in order to get D. More generally, $\coprod_i S_i \to \coprod_i D_i$ is a relative cell complex.

Example 4.30. Let $\mathcal{C} = \mathrm{Top}$, and take the cells to be the inclusion of spheres as boundaries of disks: $\{S^{k-1} \to D^k : k \geq 0\}$, where $S^{-1} \to D^0$ is $\varnothing \to *$. A cell complex is a CW-complex if

[12] As in any *cofibrantly generated* model category.

[13] You can ignore this for now if you don't know what it means.

[14] To quote the authors: "Using coproducts in the definition keeps us closer to classical cell theory, minimizes the need for set theoretic arguments, and prescribes $\mathcal{J}$-cell complexes in the form that they actually appear in all versions of the small object argument."

we restrict to attaching only cells $S^{j-1} \to D^j$ of dimension j at stage j,[15] i.e. the pushout above rather looks like

$$\begin{array}{ccc} \coprod_i S^{j-1} & \longrightarrow & \coprod_i D^j \\ \downarrow & & \downarrow \\ X^{(j-1)} & \longrightarrow & X^{(j)} \end{array}$$

In this case, $X^{(0)}$ consists of a bunch of points, then in $X^{(1)}$ we have glued some intervals by the endpoints to those points, then in $X^{(2)}$ we add some disks, etc., and then we take the colimit (the union). Note that CW-complexes do not make sense for a general $\mathcal{J}$ as "dimension" does not make sense.

Relative cell complexes are, in particular, Hurewicz cofibrations.[16]

In the case of $\mathcal{C} = \mathrm{Top}_*$, we can do the same as above with the cells $\{S_+^{k-1} \to D_+^k : k \geq 0\}$, getting *pointed cell complexes*. Note in this case the coproducts are wedges.

The Whitehead theorem, i.e. a map between CW-complexes is a weak equivalence iff it is a homotopy equivalence, generalizes to relative cell complexes in spaces, pointed or not.

Remark 4.31. One advantage of CW-complexes over general cell complexes is the possibility of doing induction arguments involving dimension. In cell complexes, the order in which we attach cells is, a priori, arbitrary, so we cannot mimic the induction arguments from CW complexes that depend on stage j having only cells of dimension at most j. Some induction arguments, those that do not depend on the dimension of cells, can still be carried out. Cellular homology is also only possible with CW-complexes.

Remark 4.32. In Top_*, consider the cells given by $\{S^{k-1} \to D^k : k \geq 1\}$, where the sphere and the disk have been endowed with basepoints and the inclusions are basepoint-preserving. This would seem to be a more natural choice of cells, but if you stop and inspect what kind of spaces you get, you'll see you only get connected ones.

We will now introduce cell spectra, but first we need a definition.

Definition 4.33. If Y is a space and $d \geq 0$, we denote by $\Sigma_+^{\infty-d}(Y)$ the spectrum $\mathrm{sh}^{-d}\Sigma_+^\infty(Y)$, and similarly for pointed spaces, without the $+$.

In other words, $\Sigma^{\infty-d}(Y)$ is the spectrum which has $*$ in degrees $0 \leq k < d$, and then has Y, ΣY, etc. Note that $\Sigma^{\infty-d} : \mathrm{Top}_* \to \mathrm{Sp}$ is a functor, and it is a left adjoint: its right adjoint is the functor $\Omega^{\infty-d} : \mathrm{Sp} \to \mathrm{Top}_*$, the d-space functor, which makes X to X_d.

The $\Sigma^{\infty-d}$ notation is justified because $\mathrm{sh} \simeq \Sigma$, as we shall see in Chapter 5. There, we will also see that (Σ, Ω) is an adjoint equivalence in Sp, in a homotopical sense.

Example 4.34. Let $\mathcal{C} = \mathrm{Sp}$, and define a cell of dimension $j \in \mathbb{Z}$ to be a map

$$\Sigma_+^{\infty-n} S^{n+j-1} \to \Sigma_+^{\infty-n} D^{n+j},$$

where $n + j \geq 0$. The additional index and the shifts back and forth (see Remark 5.3) are there to allow cells of negative dimension. We thus get the notions of *cell spectrum* and *relative cell spectrum*.

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[15] For an example of a cell complex without a CW-complex structure, see MO:23415. On the other hand, every cell complex is homotopy-equivalent to a CW-complex.

[16] An elementary proof for the fact that relative CW-complexes are Hurewicz cofibrations can be found in [**May99a**, Page 75]. A proof of the general case can probably be made in a similar fashion, or I can take out the big guns if you allow me: we know that homotopy equivalences are weak equivalences, and that Hurewicz fibrations are Serre fibrations; this means that $\mathrm{id} : \mathrm{Top}_{\mathrm{Strom}} \to \mathrm{Top}_{\mathrm{Quillen}}$ is a right Quillen functor; in particular, its left adjoint $\mathrm{id} : \mathrm{Top}_{\mathrm{Quillen}} \to \mathrm{Top}_{\mathrm{Strom}}$ is left Quillen, so it takes cofibrations to cofibrations.

Note that, if $j \geq 0$, then $D^{n+j} \cong \Sigma^n D^j$. So, cells of dimension $j \geq 0$ can be gotten by considering the space cell $S^{j-1} \to D^j$, then applying Σ^n for some $n \geq 0$, then applying $\Sigma_+^{\infty - n}$. Equivalently, you start with the cell $S^{j+n-1} \to D^{j+n}$ and apply $\Sigma_+^{\infty-n}$.

For negative j, you pick an n large enough so that $n + j \geq 0$, start with $S^{n+j-1} \to D^{n+j}$, then apply $\Sigma_+^{\infty-n}$ to it.

Considering the set of all these cells, we get the notion of a *cell spectrum*.

Here is a property which gives an alternative definition of a cell spectrum.[17]

Proposition 4.35. *A spectrum X is a cell spectrum if and only if all the structure maps $\Sigma X_n \to X_{n+1}$ as well as $* \to X_0$ are relative cell complexes (of pointed spaces).*

PROOF. ($\Rightarrow$) First, note that each cell $\Sigma_+^{\infty-n} S^{n+j-1} \to \Sigma_+^{\infty-n} D^{n+j}$ is such that each of its component maps is a relative cell complex. Indeed, the component maps are either $* \to *$ or $S_+^k \to D_+^{k+1}$. Now, you can see that this is preserved by taking pushouts, coproducts and sequential colimits starting from $X_{-1} = *$.

($\Leftarrow$) For all $n \geq 0$, define $X\langle n \rangle$ to be the spectrum such that $X\langle n \rangle_q = \begin{cases} X_q & \text{if } q \leq n \\ \Sigma^{q-n} X_n & \text{if } q > n \end{cases}$ with obvious structure maps. Define maps

$$\lambda_n : \Sigma^{\infty-n-1}\Sigma X_n \longrightarrow \Sigma^{\infty-n} X_n \longrightarrow X\langle n \rangle$$

where the first map is the transpose of the identity map $\Sigma X_n \to (\Sigma^{\infty-n} X_n)_{n+1}$ and the second map is the inclusion of the basepoint in low dimensions and the identity in high dimensions. Define maps $X\langle n \rangle \to X\langle n+1 \rangle$ by id up to level n, then $\rho_n : \Sigma X_n \to X_{n+1}$, then its suspensions. Using these maps and the $\Sigma^{\infty-n-1}\rho_n : \Sigma^{\infty-n-1}\Sigma X_n \to \Sigma^{\infty-n-1} X_{n+1}$, we have the following:

$$\begin{array}{ccccccccc}
\Sigma^{\infty-1}\Sigma X_0 & \longrightarrow & \Sigma^{\infty-1} X_1 & & \Sigma^{\infty-3}\Sigma X_2 & \longrightarrow & \Sigma^{\infty-3} X_3 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
X\langle 0 \rangle & \longrightarrow & X\langle 1 \rangle & \longrightarrow & X\langle 2 \rangle & \longrightarrow & X\langle 3 \rangle & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & & & \\
 & & \Sigma^{\infty-2}\Sigma X_1 & \longrightarrow & \Sigma^{\infty-2} X_2 & & & &
\end{array}$$

To see why those pushouts look like that, note that in level q those squares are of the form

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ X_q & \longrightarrow & X_q \end{array} \quad \text{for } q \leq n, \quad \text{and} \quad \begin{array}{ccc} \Sigma^k X_n & \xrightarrow{\Sigma^{k-1}\rho_n} & \Sigma^{k-1} X_{n+1} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \Sigma^k X_n & \xrightarrow[\Sigma^{k-1}\rho_n]{} & \Sigma^{k-1} X_{n+1} \end{array} \quad \text{for } q = n+k,\ k \geq 1.$$

Also, $\mathrm{colim}_n X\langle n \rangle$ is isomorphic to X, since indeed with n big enough, $X\langle n \rangle_q$ is just X_q, and you can check that the structure maps in the colimit are also those of X.

Since the colimit of a sequence of relative cell spectra that starts with a cell spectrum is a cell spectrum by an exercise in the sheets, then by induction we just need to observe that X_0 is a cell spectrum and that $X\langle n \rangle \to X\langle n+1 \rangle$ is a relative cell spectrum. By the same exercise, the latter reduces to seeing that $\Sigma^{\infty-n-1}\rho_n : \Sigma^{\infty-n-1}\Sigma X_n \to \Sigma^{\infty-n-1} X_{n+1}$ is a relative cell spectrum.

[17]Peter May prefers the definition of cellular spectra as we have given it above, dismissing the approach via Proposition 4.35: "In contrast with earlier treatments, our CW theory is developed on the spectrum level and has nothing whatever to do with any possible cell structures on the component spaces of spectra. I view the use of space level cell structures in this context as an obsolete historical detour that serves no useful mathematical purpose." [**May96**, Pages 122-123].

To see all this: note that if A is a cellular pointed space, then $\Sigma^{\infty-n}A$ is a cell spectrum. Indeed, just describe A as a sequential colimit of pushouts of attaching maps and apply $\Sigma^{\infty-n}$ to it: as a left adjoint, it commutes with all these colimits, and cells get taken to cells by their definition. This proves that $X\langle 0\rangle = \Sigma^\infty X_0$ is a cell spectrum.

More generally and by a similar argument, if $A \to B$ is a relative cell complex of pointed spaces, then $\Sigma^{\infty-n}A \to \Sigma^{\infty-n}B$ is a relative cell spectrum. This proves that $\Sigma^{\infty-n-1}\rho_n$ is a relative cell spectrum, thus finishing the proof. □

Example 4.36. If A is a pointed cell complex, then $\Sigma^\infty A$ is a cell spectrum. Similarly, $\Sigma^{\infty-d}A$ is a cell spectrum for all d.

Remark 4.37. If X is a cell spectrum, then all its spaces X_n are pointed cell complexes. To see this, first observe that the suspension of a pointed cell complex is a pointed cell complex, because Σ preserves colimits and takes cells to cells. Therefore, ΣX_0 is a cell complex, so X_1 is a cell complex as both maps $* \to \Sigma X_0 \xrightarrow{\rho_0} X_1$ are relative cell complexes.

Remark 4.38. The CW-spectra of [Ada74] are defined similarly to the characterization above of our cell spectra, but it's more restrictive: 1) each space X_n is required to be a CW-complex, and 2) each map $\Sigma X_n \to X_{n+1}$ is a subcomplex inclusion (more than a mere relative CW-complex), where ΣX_n has a CW-structure gotten from that of X_n where each k-cell in X_n becomes a $k+1$-cell in X_{n+1}.

In this case, we can easily talk of "cells" in X similarly to the cells of a CW-complex: if $k \in \mathbb{Z}$, a k-cell in X is a $(k+n)$-cell in X_n for each n large enough so that $k+n \geq 0$, subject to the relation that we identify it with its image $(k+n+1)$-cell in X_{n+1}. Thus, a CW-spectrum is really a CW-complex-like object where we allow cells of negative dimension. While we shall not use CW-spectra, this intuition is appealing.

Remark 4.39. In the previous proof, note that $\Sigma^{\infty-n}X_n \to X\langle n\rangle$ is a weak equivalence. On the other hand, there is no non-trivial map of spectra $\Sigma^{\infty-n}X_n \to \Sigma^{\infty-n-1}X_{n+1}$, since the maps up to level n are maps to the point, and compatibility of the structure maps would imply that the map at level $n+1$ is trivial, and so all the higher ones are trivial as well:

$$\begin{array}{ccc} \Sigma X_n & \longrightarrow & * \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ \Sigma X_n & \longrightarrow & X_{n+1} \end{array}$$

On the other hand, we have a zig-zag from $\Sigma^{\infty-n}X_n$ to $\Sigma^{\infty-n-1}X_{n+1}$; we would like to say that X is the (homotopy?) colimit of the $\Sigma^{\infty-n}X_n$. This can be made to be meaningful, see e.g. [Sch, II.5.12] for an explanation in the more complicated setting of symmetric spectra of simplicial sets. It is of theoretical relevance: it says that every spectrum can be build from desuspensions of suspension spectra. Note how stage n has trivial homotopy groups below level $-n$: each successive stage sees one more non-trivial homotopy group.

Recall that every (pointed) space X admits a (pointed) CW-approximation, i.e. a weak homotopy equivalence $X' \to X$ from a CW-complex X'. See e.g. [Hat02, 4.13] or [May99a, 10.5] for an explicit construction by attaching cells to X, or [GJ99, 2.3, 11.4] for a more roundabout (but functorial) description.

Proposition 4.40. *There is a functor $Q : \mathrm{Sp} \to \mathrm{Sp}$ with values in cell spectra, and a natural weak equivalence $QX \to X$, which actually is a level equivalence.*[18]

[18]This is an unfortunate clash of notation with the functor Q from Remark 4.23. This is definitely not the same construction. In our defense, Q is common notation for both these functors.

PROOF. We fix a functorial cell approximation functor in based spaces. In fact, we need something more general: we need a functorial way to factor a map of pointed spaces into a relative cell complex followed by a weak equivalence. See [**Ark11**, 2.4.9] for an elementary proof when the spaces are connected and a relative CW-complex is built, or [**MP12**, 17.2.2] for a more abstract proof (see also Remark 4.42). Factoring $* \to Y$ gives a natural weak equivalence $\Gamma Y \to Y$ where $\Gamma : \mathrm{Top}_* \to \mathrm{Top}_*$ has values in pointed cell complexes.

Let X be a spectrum. Define $(QX)_0$ to be ΓX_0, and $f_0 : (QX)_0 \to X_0$ to be the weak equivalence $\Gamma X_0 \to X_0$.

Define $(QX)_1$ by factoring the map

$$\Sigma(QX)_0 \xrightarrow{\Sigma f_0} \Sigma X_0 \xrightarrow{\rho_0} X_1$$

into a relative cell complex $\sigma_0 : \Sigma(QX)_0 \to (QX)_1$ followed by a weak equivalence $f_1 : (QX)_1 \to X_1$.[19]

Iterating this process defines a spectrum $QX = ((QX)_n, \sigma_n)$, which is a cell spectrum by Proposition 4.35; Q defines a functor; and the weak equivalences f_n assemble to a natural weak equivalence of spectra $f : QX \to X$. □

The functor Q is our *cofibrant replacement* functor.

Corollary 4.41. *The functor Q takes weak equivalences to weak equivalences.*

PROOF. Follows from the 2-out-of-3 property of weak equivalences, and naturality of $QX \to X$. □

Remark 4.42.

(1) In the previous section we defined a spectrum to be fibrant if it is an Ω-spectrum. In this section, we haven't defined what it means for a spectrum to be cofibrant, because we won't need this notion in full generality. But you should know that cell spectra are particular cases of cofibrant spectra.[20]

(2) In the proof above, we used a functorial factorization of a map into a relative cell complex followed by a weak equivalence. Actually, the relative cell complex can be improved to a relative CW-complex [**Hir15**]. Therefore, the same proof demonstrates that Q can be taken to have values in spectra with 0-th space a pointed CW-complex, and with relative CW-complexes as structure maps. Similarly as in Remark 4.37, these spectra then have CW-complexes at each level. However, they are not CW-spectra in the sense of Remark 4.38, because the structure maps need not be subcomplex inclusions!

The good news is that [**Hir15**] also constructs another factorization of maps $X \to Y$ similarly as above but now, if X is a CW-complex, then the map out of X is the inclusion of a subcomplex. Using this factorization, the proof above immediately gives that we can build a Q that actually takes values in CW-spectra. We shan't be using this, though.

6. Second definition of the stable homotopy category

We will now follow the plan laid out at the beginning of Section 3. Recall the cylinder and the maps $\iota_0, \iota_1 : X \to X \wedge I_+$ from Definition 3.29.

Definition 4.43. A *homotopy* of maps of spectra $X \to Y$ is a map $H : X \wedge I_+ \to Y$.

If $f = H \circ \iota_0$ and $g = H \circ \iota_1$, we say that H is a homotopy from f to g. If f and g are homotopical, we write $f \sim g$.

[19] As a side remark, note also that the map Σf_0 would be a weak equivalence if X_0 were well-pointed. We don't need this, though.

[20] More precisely, in the model category of spectra that is underlying our thoughts, the class of cofibrant spectra is the class of *retracts* of cell spectra.

A *homotopy equivalence* is a map $f : X \to Y$ for which there exists a map $g : Y \to X$, a homotopy between $g \circ f$ and id_X, and a homotopy between $f \circ g$ and id_Y.

Remark 4.44.

(1) A based homotopy of spaces is given by a map of spaces $H : X \times I \to Y$ such that $H(x_0, t) = y_0$ for all $t \in I$; by adjunction, this is equivalently a map of based spaces $X \wedge I_+ \to Y$.

(2) A homotopy of maps of spectra is equivalently given by a sequence of based homotopies of spaces $H_n : X_n \wedge I_+ \to Y_n$ such that the following squares commute:

$$\begin{array}{ccc} \Sigma X_n \wedge I_+ & \xrightarrow{\Sigma H_n} & \Sigma Y_n \\ {\scriptstyle \rho_n^X \wedge \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \rho_n^Y} \\ X_{n+1} \wedge I_+ & \xrightarrow[H_{n+1}]{} & Y_{n+1}. \end{array}$$

More sloppily, but in words, we might say that "H_{n+1} is a homotopy relative to ΣH_n".

(3) By adjunction, a homotopy is equivalently a map $X \to F(I_+, Y)$ in Sp or a map $I_+ \to \mathrm{Map}(X, Y)$ in Top_* or a map $I \to \mathrm{Map}(X, Y)$ in Top, i.e. a path between maps. So two maps are homotopic iff they lie in the same path component of $\mathrm{Map}(X, Y)$.

(4) Homotopic maps induce the same map in homotopy groups, since the analogous statement is true in spaces. In particular, if f is a homotopy equivalence then it is a weak equivalence.

(5) If a functor out of Sp takes weak equivalences to isomorphisms, then it takes homotopical maps to equal maps. This is proven analogously to Remark 4.5.

Exercise 4.45. *Prove that homotopy is an equivalence relation on* $\mathrm{Hom}_{\mathrm{Sp}}(X, Y)$.

We denote by $[X, Y]$ the set of homotopy classes of maps $X \to Y$.

Remark 4.46. Homotopy equivalence of spectra is, in full generality, not the relation we are interested in. Consider $\mathbb{S}'$: this is the sphere spectrum except in degree 0 it is *, as in Example 4.18. Then any map $\mathbb{S} \to \mathbb{S}'$ is constant at each level: by definition at level 0, and forced by the compatibility with the structure maps in higher levels. So there can be no homotopy equivalence between $\mathbb{S}$ and $\mathbb{S}'$, though they morally should be equivalent in some way, and indeed they are weakly equivalent.

Proposition 4.47. *If $f : X \to Y$ is a level equivalence of cell spectra, then it is a homotopy equivalence.*

PROOF. The maps $f_n : X_n \to Y_n$ are weak equivalences of cell complexes (using Remark 4.37), so by the Whitehead theorem, they are homotopy equivalences. A homotopy of maps of spectra requires relative homotopies, so we need to work a bit harder. We are in the situation of commutative diagrams:

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow[\Sigma f_n]{\sim} & \Sigma Y_n \\ \downarrow & & \downarrow \\ X_{n+1} & \xrightarrow[f_{n+1}]{\sim} & Y_{n+1}. \end{array}$$

The two vertical maps are relative cell complexes, hence cofibrations. The two horizontal maps are homotopy equivalences. By [May99a, Page 47], the homotopy between the identity and the composition of f_{n+1} and its homotopy inverse is actually a homotopy H_{n+1} relative to ΣH_n. □

Theorem 4.48 (Whitehead)**.** *If $f : X \to Y$ is a weak equivalence of cell Ω-spectra, then it is a homotopy equivalence.*

PROOF. Since X and Y are Ω-spectra, then by Proposition 4.20 f is a level equivalence. Since X and Y are cell spectra, then by Proposition 4.47 f is a homotopy equivalence. □

Remark 4.49. If X is an Ω-spectrum, then QX is an Ω-spectrum as well. This follows from the fact that $f : QX \to X$ is a level equivalence (Proposition 4.40): we have the following commutative squares,

$$\begin{array}{ccc} (QX)_n & \xrightarrow[\sim]{f_n} & X_n \\ {\scriptstyle \widetilde{\sigma_n}}\downarrow & & \downarrow{\scriptstyle \sim\ \widetilde{\rho_n}} \\ \Omega(QX)_{n+1} & \xrightarrow[\Omega f_{n+1}]{\sim} & \Omega X_{n+1} \end{array}$$

so $\widetilde{\sigma_n}$ is a weak equivalence for all n.

In particular, for any spectrum X, QRX is a cell Ω-spectrum.[21]

Definition 4.50. Let $\mathrm{Ho}(\mathrm{Sp})$ be the (locally small) category with objects those of Sp, and with arrows from X to Y given by the set $[QRX, QRY]$. Define a functor $\gamma : \mathrm{Sp} \to \mathrm{Ho}(\mathrm{Sp})$ which is the identity on objects, and which takes $f : X \to Y$ to the homotopy class of $QRf : QRX \to QRY$.

Exercise 4.51. *Prove that* $\mathrm{Ho}(\mathrm{Sp})$ *is well-defined. Among other things, you need to prove that composition does not depend on the choice of representative for a homotopy class. So you will need to prove that if you have maps in* Sp

$$A \xrightarrow{a} X \underset{f'}{\overset{f}{\rightrightarrows}} Y \xrightarrow{b} B$$

then $f \sim f'$ *implies* $b \circ f \sim b \circ f'$ *and* $f \circ a \sim f' \circ a$.

Lemma 4.52. *(1) The functor* $\gamma : \mathrm{Sp} \to \mathrm{Ho}(\mathrm{Sp})$ *is such that the function*

$$\gamma : \mathrm{Hom}_{\mathrm{Sp}}(X, Y) \to \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp})}(X, Y)$$

is surjective if X, Y *are cell* Ω*-spectra, and in this case it factors as*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Sp}}(X, Y) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp})}(X, Y) \\ & \searrow \quad\quad \nearrow_{\cong} & \\ & [X, Y] & \end{array}$$

(2) $\gamma(f)$ *is an isomorphism in* $\mathrm{Ho}(\mathrm{Sp})$ *if and only if* f *is a weak equivalence in* Sp.
(3) Every morphism of $\mathrm{Ho}(\mathrm{Sp})$ *is a composite of morphisms of the form* $\gamma(a)$ *or* $\gamma(w)^{-1}$ *for* w *a weak equivalence.*

PROOF. (1) Let X, Y be cell Ω-spectra. Let $[f : QRX \to QRY] \in \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp})}(X, Y)$ be a morphism. We have the following solid diagram in Sp

$$\begin{array}{ccc} QRX & \xrightarrow{f} & QRY \\ \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\ QX & & QY \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ X & \underset{g}{\dashrightarrow} & Y \end{array}$$

[21]If you're wondering whether RQX is also a cell Ω-spectrum, then I'll have to disappoint you, it isn't in general. The functor R may not take cell spectra to cell spectra. You could build an R' that would satisfy this extra condition, but it wouldn't be as explicit as our R.

where the arrows marked $\sim$ are weak equivalences between cell Ω-spectra, so they are homotopy equivalences; inverting the two pointing backwards, we can define the dotted map g. Now, the two maps $f, QRg : QRX \to QRY$ are homotopical. Indeed, after composing with the homotopy equivalences $X \to QRX$ on the left and $QRY \to Y$ on the right, f becomes g by definition, whereas QRg becomes homotopical to g, using naturality of Q and R. So g is the map we were looking for.

Moreover, if $f \sim f'$, then by construction $g \sim g'$, too, so the second statement follows.

(2) If $\gamma(f) = [QRf]$ is an isomorphism, that means QRf is a homotopy equivalence, so it is also a weak equivalence. Hence the top arrow in the following commutative diagram is a weak equivalence,

$$\begin{array}{ccc} QRX & \xrightarrow{QRf} & QRY \\ \uparrow\sim & & \uparrow\sim \\ QX & \xrightarrow{Qf} & QY \\ \downarrow\sim & & \downarrow\sim \\ X & \xrightarrow[f]{} & Y \end{array}$$

so by 2-out-of-3, Qf is also a weak equivalence, and again by 2-out-of-3, f is a weak equivalence.

Conversely, if $f : X \to Y$ is a weak equivalence, then QRf is a weak equivalence between cell Ω-spectra, so it's a homotopy equivalence, i.e. $\gamma(f)$ is an isomorphism.

(3) Let $f : \gamma(X) \to \gamma(Y)$ in $\mathrm{Ho}(\mathrm{Sp})$ (we stick to the notation $\gamma(X)$ for the sake of clarity). We have $X \xleftarrow{\sim} QX \xrightarrow{\sim} QRX$ in Sp, and similarly for Y. We now apply γ to these, so we get the solid diagram

$$\begin{array}{ccccc} \gamma(X) & \xleftarrow{\cong} & \gamma(QX) & \xrightarrow{\cong} & \gamma(QRX) \\ {\scriptstyle f}\downarrow & & & & \vdots \\ \gamma(Y) & \xleftarrow[\cong]{} & \gamma(QY) & \xrightarrow[\cong]{} & \gamma(QRY) \end{array} \tag{4.53}$$

where the horizontal maps are isomorphisms by the previous part of this lemma. Since QRX and QRY are cell Ω-spectra, then by part (1) we know that the dotted map $\gamma(QRX) \to \gamma(QRY)$ (defined as the composition) is of the form $\gamma(f')$ for an $f' \in \mathrm{Hom}_{\mathrm{Sp}}(QRX, QRY)$. The commutativity of the rectangle gives our decomposition of f.

□

Theorem 4.54. *There is a unique isomorphism of categories* $T : \mathrm{Sp}[\mathcal{W}^{-1}] \to \mathrm{Ho}(\mathrm{Sp})$ *such that* $T \circ \iota = \gamma$.

PROOF. We will prove that $\gamma : \mathrm{Sp} \to \mathrm{Ho}(\mathrm{Sp})$ satisfies the universal property of Proposition 4.1.

First, we need to verify that γ takes weak equivalences to isomorphisms; we just did this in Lemma 4.52.(2).

Now, let $\mathcal{D}$ be a category with a functor $F : \mathrm{Sp} \to \mathcal{D}$ which sends weak equivalences to isomorphisms. We need to prove there exists a unique functor F' making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Sp} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle\gamma}\downarrow & \nearrow_{F'} & \\ \mathrm{Ho}(\mathrm{Sp}). & & \end{array}$$

Let us first prove that there can only be one such F'. Since γ is the identity on objects, we need to set $F'X = FX$. By Lemma 4.52.(3), we know that any map in $\mathrm{Ho}(\mathrm{Sp})$ can be written as a composition of maps of the form $\gamma(a)$ or $\gamma(w)^{-1}$ for w a weak equivalence. Now, since $F' \circ \gamma = F$, then $F'(\gamma(a)) = F(a)$, and $F'(\gamma(w)^{-1}) = F'(\gamma(w))^{-1} = F(w)^{-1}$. This proves that F' is unique, if it exists.

To define F', we set $F'(X) = X$, and for $F'(f : X \to Y)$ we use the decomposition from (4.53), so we define it as the composition in the following diagram, where the horizontal maps are of the form $F(w)$ for w a weak equivalence in Sp.

$$\begin{array}{ccccc} F(X) & \xleftarrow{\cong} & F(QX) & \xrightarrow{\cong} & F(QRX) \\ {\scriptstyle F'(f)}\downarrow & & & & \downarrow{\scriptstyle F(f')} \\ F(Y) & \xleftarrow[\cong]{} & F(QY) & \xrightarrow[\cong]{} & F(QRY) \end{array}$$

We need to check that the definition of $F'(f)$ doesn't depend on the choice of f'. If $f'' : QRX \to QRY$ is another map such that $\gamma(f'')$ makes (4.53) commute, then $f'' \sim f'$ by the second part of Lemma 4.52.(1). But then, since γ takes weak equivalences to isomorphisms, then by Remark 4.44.5 it also takes homotopical maps to equal maps, so $\gamma(f'') = \gamma(f')$ and F' is well-defined. It is not hard to check that it preserves identities and compositions. □

From the above theorem, we can deduce a couple of properties of the localization.

Corollary 4.55. *(1) The category* $\mathrm{Sp}[\mathcal{W}^{-1}]$ *is locally small.*

(2) A morphism in Sp *is inverted by* $\mathrm{Sp} \to \mathrm{Sp}[\mathcal{W}^{-1}]$ *if and only if it is a weak equivalence.*[22]

(3) A functor $F : \mathrm{Sp} \to \mathrm{Sp}$ *that preserves weak equivalences descends uniquely to a functor* $\mathrm{Ho}(F) : \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ho}(\mathrm{Sp})$*, often simply denoted by* F*. If* F *has a right adjoint* G *that preserves weak equivalences, then* $(\mathrm{Ho}(F), \mathrm{Ho}(G))$ *is an adjoint equivalence.*[23]

PROOF. (1) This follows from $\mathrm{Ho}(\mathrm{Sp})$ being locally small.

(2) This follows from the analogous property for $\gamma : \mathrm{Sp} \to \mathrm{Ho}(\mathrm{Sp})$, which was proven in Lemma 4.52.2.

(3) This follows from Proposition 4.3 and Exercise 4.7. □

Remark 4.56. The following is useful; we prefer to omit the proof. For abstract formulations and proofs, see [**MP12**, Chapter 16] or [**Hov99**, 1.3.2].

(1) Let $F : \mathrm{Top}_* \to \mathrm{Sp}$ be a functor. Suppose that F takes cell complexes to cell spectra, and it takes weak equivalences between cell complexes to weak equivalences. Then it induces a functor $\mathbb{L}F : \mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{Ho}(\mathrm{Sp})$, defined by first replacing a pointed space by a pointed cell complex, then applying F. Here $\mathrm{Ho}(\mathrm{Top}_*)$ is $\mathrm{Top}_*[\mathcal{W}^{-1}]$ where $\mathcal{W}$ are the weak equivalences.

(2) If $G : \mathrm{Sp} \to \mathrm{Top}_*$ takes weak equivalences between Ω-spectra to weak equivalences, then it induces a functor $\mathbb{R}G : \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ho}(\mathrm{Top}_*)$ defined by first replacing a spectrum by an Ω-spectrum, then applying G.

[22] This property of the class of arrows $\mathcal{W}$ is called *saturation*.

[23] The same is true if we replace Sp by Top or Top_*, in the domain or in the codomain.

(3) If F, G are above are such that (F, G) is an adjoint pair, then $(\mathbb{L}F, \mathbb{R}G)$ is an adjoint pair.
(4) Using the above, one can construct the adjoint pair $(\mathbb{L}\Sigma^\infty, \mathbb{R}\Omega^\infty)$. This $\mathbb{R}\Omega^\infty$ is what many homotopy theorists mean when when they write Ω^∞ (recall Remark 3.19). The notation gains meaning: $\mathbb{R}\Omega^\infty(X)$ is the zero-space of the Ω-spectrum RX, hence an honest infinite loop space.

The definition of the maps in $\mathrm{Ho}(\mathrm{Sp})$ is a bit complicated, but it has the advantage that it's then easy to define a functor $\mathrm{Sp} \to \mathrm{Ho}(\mathrm{Sp})$. We can simplify the description of maps in the stable homotopy category a bit:

Exercise 4.57. *(1) Prove that* $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y) \cong [QX, RY]$. [24]
(2) If X is a cell spectrum, prove that $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y) \cong [X, RY]$.
(3) If Y is an Ω-spectrum, prove that $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y) \cong [QX, Y]$.
(4) If X is a cell spectrum and Y is an Ω-spectrum, prove that $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y) \cong [X, Y]$.

Remark 4.58. For the purposes of this course, we will call $\mathrm{Ho}(\mathrm{Sp})$ *the stable homotopy category*. In common mathematical practice, any category equivalent to $\mathrm{Ho}(\mathrm{Sp})$ is called like that. From Side remark 3.17 we know there are many different but equivalent ways of producing one. For a classical axiomatic presentation of it, see [**Mar83**, Section 2.1]. See also Side remark 6.12 for a development of this idea.

Example 4.59. In Example 4.17 we talked about the degree p map and the Hopf map, and we observed that we couldn't define them as maps of spectra with the strict definition. In Example 4.26 we defined a stable version of the Hopf map as a map $\eta : R\Sigma\mathbb{S} \to R\mathbb{S}$.

By Exercise 4.57 we know we can give a representative of $\gamma(\eta)$ in Sp as a map $\Sigma\mathbb{S} \to R\Sigma\mathbb{S}$, without fibrant-replacing on the domain, since $\Sigma\mathbb{S}$ is cell. In this case, using compactness of S^1, we can actually be very explicit about it, without using the exercise.

By adjunction, to determine a map $\eta : \Sigma\mathbb{S} = \Sigma^\infty S^1 \to R\mathbb{S}$ we may equivalently determine a map of based spaces $S^1 \to (R\mathbb{S})_0$. Recall that $(R\mathbb{S})_0$ is the telescope

$$(R\mathbb{S})_0 = \mathrm{hocolim}(S^0 \to \Omega S^1 \to \Omega^2 S^2 \to \cdots)$$

so by the results at the end of Section 1.5, $\mathrm{Map}_{\mathrm{Top}_*}(S^1, (R\mathbb{S})_0) \cong \mathrm{colim}_n \mathrm{Map}_{\mathrm{Top}_*}(S^1, \Omega^n S^n)$. To give an object in this colimit it is enough to give a map $S^1 \to \Omega^m S^m$ for some m. We choose $m = 2$: we let the map $S^1 \to \Omega^2 S^2$ be the transpose to the Hopf map $S^3 \to S^2$.

Compare to Example 4.18, where we showed how to define a degree p map $\mathbb{S} \to \mathbb{S}$ in $\mathrm{Sp}[\mathcal{W}^{-1}]$. Similarly, we could have directly defined the Hopf map in $\mathrm{Sp}[\mathcal{W}^{-1}]$ easily, and similarly as above, we could define the degree p map as an object in $[\mathbb{S}, R\mathbb{S}]$.

Let us end with yet another remark about how to see "eventually-defined maps of spectra" as actual maps.

Remark 4.60. Let X and Y be spectra. Let $k \geq 1$, and let $f_n : X_n \to Y_n$ for $n \geq k$ be maps that commute with the structure maps. At the end of Example 4.26 we defined a map $\varphi : RX \to RY$. So $[Q\varphi]$ is a map $X \to Y$ in $\mathrm{Ho}(\mathrm{Sp})$. We can construct a representative for it more directly if we add some hypotheses.

(1) If X is cell and Y is an Ω-spectrum, then by Lemma 4.52.(1) we can define a unique-up-to-homotopy map $f : X \to Y$ such that QRf is the same as $Q\varphi$ in $\mathrm{Ho}(\mathrm{Sp})$. They both represent the original "eventually-defined map".

[24] In general, this follows from [**Hov99**, 1.2.5.iv], of which one of the two dual versions adapted to this case says that if A is a cell spectrum and $f : B \to B'$ a weak equivalence between Ω-spectra, then $f_* : [A, B] \to [A, B']$ is an isomorphism.

Here's an alternative way to go about it. We use the following general result: if $A \to B$ is a weak equivalence of spaces and $U \to V$ is a relative cell complex, then, there exists a dotted map that makes the following diagram commute up to homotopy:

$$\begin{array}{ccc} U & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow \sim \\ V & \longrightarrow & B \end{array}$$

For example, for the first step, $* \to X_0$ is a relative cell complex, and you lift the map $X_0 \to \Omega Y_0 \to \Omega Y_1$ against the weak equivalence $Y_0 \to \Omega Y_1$.

(2) If Y is an Ω-spectrum and Y_i is a CW-complex for $i \leq k$, you can also define such a map $X \to Y$, but now the lifting is easier because you can actually invert the weak equivalences. You use two ingredients here: first, the classical theorem that the loop space of a CW-complex has the homotopy type of a CW-complex, and a version of the Whitehead theorem that says that a weak equivalence between spaces of the homotopy type of CW-complexes is a homotopy equivalence [**MP12**, 17.3.4(i)].

CHAPTER 5

Stability results

1. Comparison of different suspensions

In Section 2.3 we talked about $\mathrm{sh}(X)$, $X \wedge S^1$ and ΣX. We will now see they are all equivalent, homotopically.

Proposition 5.1. *Let X be a spectrum. There are natural isomorphisms*

$$\pi_{n+1}(\Sigma X) \cong \pi_n(X) \cong \pi_{n-1}(\Omega X).$$ [1]

In particular, if $f : X \to X'$ be a map of spectra, then f is a weak equivalence if and only if Σf is a weak equivalence[2]*, if and only if $\Sigma^k f$ is a weak equivalence for all $k \in \mathbb{N}$; similarly, f is a weak equivalence if and only if Ωf is a weak equivalence, if and only if $\Omega^k f$ is a weak equivalence for all $k \in \mathbb{N}$.*

PROOF. By naturality, we have commutative squares

$$\begin{array}{ccc} S^1 \wedge X_0 & \xrightarrow{\eta} & \Omega(S^1 \wedge S^1 \wedge X_0) \\ {\scriptstyle \rho_0}\downarrow & & \downarrow{\scriptstyle \Omega(\mathrm{id}\wedge\rho_0)} \\ X_1 & \xrightarrow[\eta]{} & \Omega(S^1 \wedge X_1) \end{array}$$

where η is the unit of the (Σ, Ω) adjunction. So we have the following commutative diagram

$$\begin{array}{ccccccccc} \pi_{n+1}(S^1 \wedge X_0) & \xrightarrow{\eta_*} & \pi_{n+2}(S^1 \wedge S^1 \wedge X_0) & \xrightarrow{(\mathrm{id}\wedge\rho_0)_*} & \pi_{n+2}(S^1 \wedge X_1) & \xrightarrow{\eta_*} & \pi_{n+3}(S^1 \wedge S^1 \wedge X_1) & \longrightarrow & \cdots \\ & \searrow^{\mathrm{id}} & & & & \searrow^{\mathrm{id}} & & & \\ \pi_n(X_0) & \xrightarrow{\eta_*} & \pi_{n+1}(S^1 \wedge X_0) & \xrightarrow{(\rho_0)_*} & \pi_{n+1}(X_1) & \xrightarrow{\eta_*} & \pi_{n+2}(S^1 \wedge X_1) & \longrightarrow & \cdots \end{array}$$

The colimit of the first line computes $\pi_{n+1}(\Sigma X)$, and the one of the second arrow commutes $\pi_n(X)$. The vertical arrows define the desired natural isomorphism between the colimits.

The proof of the natural isomorphism $\pi_n(X) \cong \pi_{n-1}(\Omega X)$ is similar, and left as an exercise.

The last statement follows directly from the naturality squares. $\square$

We can now prove that Σ and Ω are "inverses up to equivalence" in Sp:

Corollary 5.2. *(1) The unit $X \to \Omega\Sigma X$ and the counit $\Sigma\Omega X \to X$ are stable equivalences. In particular, the adjunction (Σ, Ω) descends to an adjoint equivalence $\mathrm{Ho}(\mathrm{Sp}) \underset{\Omega}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathrm{Ho}(\mathrm{Sp})$.*

As a consequence, we will use the notation Σ^{-1} for Ω.

(2) The natural map $\Sigma X \to \mathrm{sh}(X)$ from Section 2.3 is a stable equivalence, so we have natural weak equivalences $\Sigma^n X \to \mathrm{sh}^n(X)$ for $n \geq 0$. Dually, the natural map $\mathrm{sh}^{-1}(X) \to \Omega X$ is a stable equivalence, and we have natural weak equivalences $\mathrm{sh}^{-n} X \to \Omega^n X$ for $n \geq 0$.

---

[1] Note how false this is for spaces!

[2] Note how the "well-pointed" hypothesis, needed for the true implication for pointed spaces, is gone here.

PROOF. (1) Composing the isomorphisms $\pi_n(X) \cong \pi_{n+1}(\Sigma X) \cong \pi_n(\Omega\Sigma X)$ and looking at their definitions, we see that they are induced by the unit map, and similarly for the counit. That Σ and Ω descend to the homotopy category as an adjoint equivalence follows from Corollary 4.55.(3).

(2) The map $\pi_n(\Sigma X) \to \pi_n(\mathrm{sh}(X))$ factors as $\pi_n(\Sigma X) \xrightarrow{\cong} \pi_{n-1}(X) \xrightarrow{\cong} \pi_n(\mathrm{sh}(X))$, where the first isomorphism was constructed in the proof of the proposition, and the second isomorphism is constructed similarly. $\square$

Remark 5.3. In particular, for any $d \geq 0$ and any $Y \in \mathrm{Top}_*$ we have a weak equivalence (and hence an isomorphism in $\mathrm{Ho}(\mathrm{Sp})$)

$$\Sigma^{\infty-d}Y = \mathrm{sh}^{-d}\Sigma^\infty Y \xrightarrow{\sim} \Omega^d\Sigma^\infty Y = \Sigma^{-d}\Sigma^\infty Y$$

as was to be expected. Similarly as in Remark 3.25, we have

$$\Sigma^{\infty-d}Y \cong \Sigma^{\infty-d}S^0 \wedge Y \xrightarrow{\sim} \Sigma^{-d}\Sigma^\infty S^0 \wedge Y = \Sigma^{-d}\mathbb{S} \wedge Y.$$

As for spaces which are suspensions, we have a zig-zag of weak equivalences

$$\Sigma^{\infty-n}\Sigma^n Y \xrightarrow{\sim} \Omega^n\Sigma^\infty\Sigma^n Y \cong \Omega^n\Sigma^n\Sigma^\infty Y \xleftarrow{\sim} \Sigma^\infty Y,$$

where the isomorphism comes from Exercise 3.35.(2) and the last weak equivalence comes from composing loops of the weak equivalences $X \to \Omega\Sigma X$.

Example 5.4. Let's see how does the "degree p" map now look like, using these weak equivalences. In Example 4.18 we constructed it as a map $\mathbb{S} \to \mathbb{S}$, which by inspection is exactly the map $\Sigma^{\infty-1}f_p : \Sigma^{\infty-1}S^1 \to \Sigma^{\infty-1}S^1$ where $f_p : S^1 \to S^1$ is a degree p map. We have the following solid commutative diagram,

$$\begin{array}{ccccccc}
\mathbb{S} = \Sigma^\infty S^0 & \xrightarrow{\sim} & \Omega\Sigma\Sigma^\infty S^0 & \xrightarrow{\cong} & \Omega\Sigma^\infty\Sigma S^0 & \xleftarrow{\sim} & \Sigma^{\infty-1}\Sigma S^0 \\
\vdots & & & & \downarrow{\scriptstyle \Omega\Sigma^\infty f_p} & & \downarrow{\scriptstyle \Sigma^{\infty-1}f_p} \\
\mathbb{S} = \Sigma^\infty S^0 & \xrightarrow[\sim]{} & \Omega\Sigma\Sigma^\infty S^0 & \xrightarrow[\cong]{} & \Omega\Sigma^\infty\Sigma S^0 & \xleftarrow[\sim]{} & \Sigma^{\infty-1}\Sigma S^0
\end{array}$$

The vertical dotted map does not exist in Sp, but does exist in $\mathrm{Ho}(\mathrm{Sp})$, since then we can invert the weak equivalences and just go around the diagram. We could do something similar for the Hopf map.

We saw in Section 2.3 that there is no natural map $X \wedge S^1 \to \mathrm{sh}(X)$ (or the other way around). Similarly, there is no natural map $\Sigma X \to X \wedge S^1$ (or the other way around). It would be, on stage n, a map $S^1 \wedge X_n \to X_n \wedge S^1$. The only candidate would be the twist τ. But this would not give a map of spectra, since the following diagram doesn't commute:

$$\begin{array}{ccc}
S^1 \wedge S^1 \wedge X_n & \xrightarrow{\mathrm{id}\wedge\tau} & S^1 \wedge X_n \wedge S^1 \\
{\scriptstyle \mathrm{id}\wedge\rho_n}\downarrow & & \downarrow{\scriptstyle \rho_n\wedge\mathrm{id}} \\
S^1 \wedge X_{n+1} & \xrightarrow[\tau]{} & X_{n+1} \wedge S^1
\end{array}$$

You could think that we can fix this if our structure maps for ΣX had twists in them, so that the diagram above would look like

$$\begin{array}{ccc}
S^1 \wedge S^1 \wedge X_n & \xrightarrow{\mathrm{id}\wedge\tau} & S^1 \wedge X_n \wedge S^1 \\
\downarrow{\scriptstyle \tau\wedge\mathrm{id}} & \nearrow{\scriptstyle \tau_{S^1,S^1\wedge X_n}} & \\
S^1 \wedge S^1 \wedge X_n & & \downarrow{\scriptstyle \rho_n\wedge\mathrm{id}} \\
\downarrow{\scriptstyle \mathrm{id}\wedge\rho_n} & & \\
S^1 \wedge X_{n+1} & \xrightarrow[\tau]{} & X_{n+1} \wedge S^1
\end{array}$$

where the diagonal arrow helps us prove commutativity: the bottom part commutes by naturality of τ, and the upper part by inspection.[3]

Note that the arrow $\tau : S^1 \wedge S^1 \to S^1 \wedge S^1$ is not the identity: it is a map of degree -1, so there's no contradiction with this diagram commuting and the previous one not, they're honestly different diagrams.

You could now propose that we define $\Sigma' X$ so as to include that twist. If we do that, then something else breaks: we get no natural map $\Sigma' X \to \mathrm{sh}(X)$ or $\Sigma' X \to \Sigma X$. So we are not gaining much: we still cannot connect all of the actors together easily. Let us dump this idea.

However, looking again at the diagram above: suppose the first S^1 was an S^2. In this case, we could use the fact that the twist map $S^n \wedge S^m \to S^m \wedge S^n$ has degree $(-1)^{nm}$. Then $\tau \wedge \mathrm{id} : S^2 \wedge S^1 \wedge X_n \to S^1 \wedge S^2 \wedge X_n$ is homotopic to the identity. So suppose we defined "S^2-spectra" as sequences $(X_0, X_2, X_4, \dots)$ with structure maps $S^2 \wedge X_n \to X_{n+2}$. Intuitively, this should be the same thing as spectra, because the inclusion of even natural numbers into natural numbers is cofinal. One can use these ideas to formally prove that $X \wedge S^1$ is naturally isomorphic to ΣX in $\mathrm{Ho}(\mathrm{Sp})$. See e.g. 3.22 of nLab:model struture on topological sequential spectra for a proof along these lines. We prefer not to get into that, so we will just state:

Proposition 5.5. *The functor* $- \wedge S^1$ *passes to the homotopy category:* $- \wedge S^1 : \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ho}(\mathrm{Sp})$, *and it is naturally isomorphic to* Σ *there.*

Exercise 5.6. *State the precise relationship between* ΩX, $\mathrm{sh}^{-1}(X)$ *and* $F(S^1, X)$.

2. (Co)fiber sequences

The following definitions are analogous to the definitions for spaces.

Definition 5.7. (1) The *cone* of a spectrum X is $X \wedge I$. It comes with a canonical map $X \to X \wedge I$. In level n, it is the reduced cone of X_n.

(2) The *homotopy cofiber* (or *mapping cone*) of a map $f : X \to Y$ of spectra is the spectrum Cf defined as the following pushout.

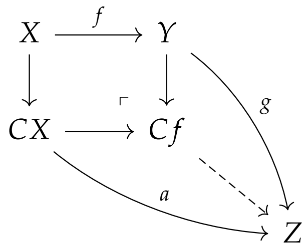
$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & \ulcorner & \downarrow \\
CX & \longrightarrow & Cf.
\end{array}$$

In level n, it is the reduced homotopy cofiber of f.

Exercise 5.8. *Let* $f : X \to Y$ *be a map of pointed spaces or of spectra. Prove that it is (based) nullhomotopic if and only if it factors via the canonical inclusion* $X \to CX$.

[3] Or, killing a flea with a steamroller, by the fact that Top_* is a symmetric monoidal category, and the coherence theorem for those.

Therefore, if we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, to determine a map a making the following solid diagram commute, is equivalent to determining a nullhomotopy of $g \circ f$.



This gives us the justification to make the following definition.

Definition 5.9. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of maps of spectra, and fix a nullhomotopy of $g \circ f$. The sequence is a *(homotopy) cofiber sequence* if the induced map $Cf \to Z$ is a weak equivalence. A *map of cofiber sequences* consists of three vertical maps making the two squares commute.

Example 5.10. If $X \to Y \to Z$ is a cofiber sequence of pointed spaces, then $\Sigma^\infty X \to \Sigma^\infty Y \to \Sigma^\infty Z$ is a cofiber sequence of spectra, and similarly for $\Sigma^{\infty - d}$.

Dually, we can define the path space PX, the homotopy fiber (mapping path spectrum) Pf, and (homotopy) fiber sequences.

In Top_*, fiber sequences have long exact sequences of homotopy groups. It is also the case in spectra:

Proposition 5.11. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a fiber sequence of spectra. It induces a natural long exact sequence of abelian groups*

$$\cdots \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Z) \xrightarrow{\partial} \pi_{k-1}(X) \longrightarrow \cdots .$$

Proof. Levelwise this is true, and then it suffices to pass to the colimit. □

More interestingly, *cofiber* sequences also give long exact sequences in homotopy! This is not true in spaces, at least not without connectivity hypotheses as in Corollary 2.29.

Proposition 5.12. *Let $X \to Y \to Z$ be a cofiber sequence of spectra. It induces a natural long exact sequence of abelian groups*

$$\cdots \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Z) \xrightarrow{\partial} \pi_{k-1}(X) \longrightarrow \cdots .$$

Proof. First, note that we can build a sequence of spectra similar to the Puppe sequence, where each two consecutive maps is a homotopy cofiber sequence. Indeed, we have levelwise Puppe sequences for each component pointed space, and we can connect them via the structure maps, getting a sequence of maps of spectra[4]

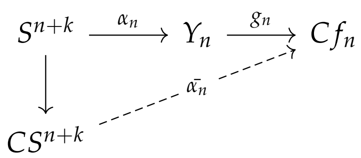
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{q} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \longrightarrow \cdots$$

where each two consecutive maps is a homotopy cofiber sequence. Hence, by Proposition 5.1, it suffices to check that $X \to Y \to Z$ induces an exact sequence of abelian groups $\pi_k X \xrightarrow{f_*} \pi_k Y \xrightarrow{g_*} \pi_k Z$ for each $k \in \mathbb{Z}$. Without loss of generality, we may assume that $Y \to Z$ is the canonical map $g : Y \to Cf$.

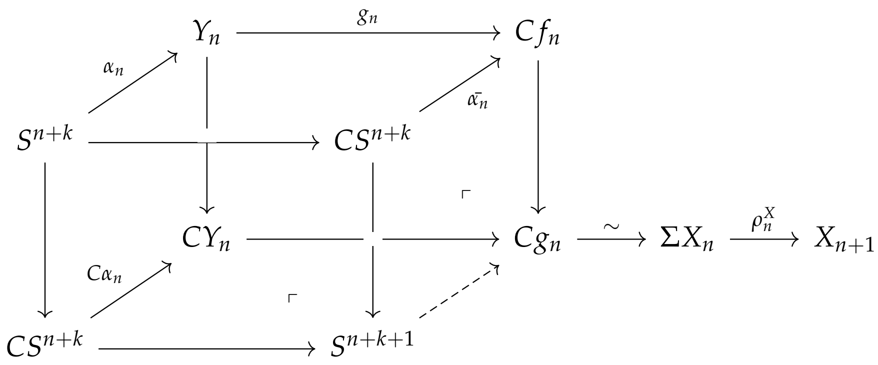
[4] For the moment we don't know what's the opposite of a map of spectra; we can take $-\Sigma f$ here to mean simply the map that has $-\Sigma f_n$ at each stage.

Now, since $g \circ f$ is nullhomotopic, then $g_* \circ f_* = 0$. Given $\alpha \in \pi_k(Y)$ such that $g_*(\alpha) = 0$, we need to see it's in the image of f_*. Fix such an α.

Since $g_*(\alpha) = 0$, there exists an n and $\alpha_n \in \pi_{n+k}(Y_n)$, a representative of α, such that $(g_n)_*(\alpha_n) = 0$ in $\pi_{n+k}(Cf_n)$. In other words, $g_n \circ \alpha_n$ is nullhomotopic, so we have an extension to the cone as in the following diagram.



By naturality of pushouts, we have the dotted map in the following diagram where the front and the back are pushouts.

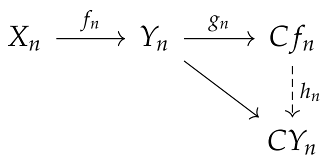


Here the weak equivalence $Cg_n \to \Sigma X_n$ comes from Puppe. We let

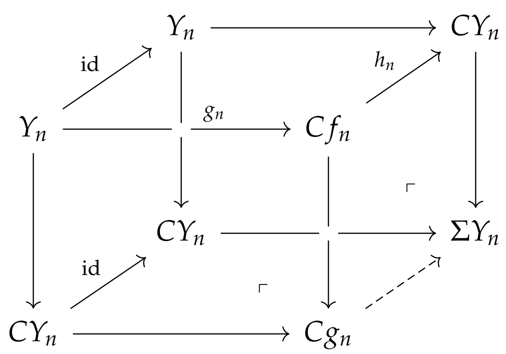
$$S^{n+k+1} \to Cg_n \to \Sigma X_n \to X_{n+1}$$

be denoted by β_{n+1}, and we let $\beta \in \pi_k(X)$ denote its associated stable homotopy class. Now want to prove that $f_*(\beta) = \alpha$.[5] [6]

Define h_n to be the following dotted map, gotten from the universal property of the homotopy cofiber, since CY_n is contractible.



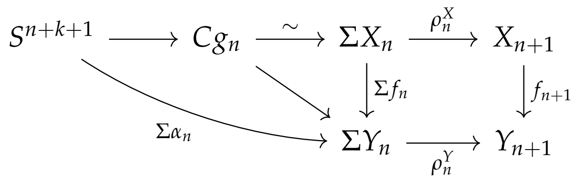
We use it to get another induced map of pushouts:



[5] I don't wish to dwell on the sign issue because it's a non-issue here: if we find a β such that $f_*(\beta) = \pm\alpha$, we're done, because if it's negative we can just take $-\beta$. Perhaps the β we've built gives $-\alpha$; allow me to be lazy and skip the check.

[6] Note how α is represented in level n and β is represented in level $n+1$. It was to be expected that β should be represented by a "later" homotopy class, since the result is false for spaces.

Now, $h_n \circ \bar{\alpha}_n = C\alpha_n$ (check!). So when you compose the two dotted arrows in the two cubes, putting the second cube behind the first one, you are actually getting $\Sigma\alpha_n : S^{n+k+1} \to \Sigma Y_n$. We have the following commutative diagram:



Now, if we go right and then down, we get $f_{n+1}(\beta_{n+1})$. If we go down with the curved arrow then right, we get $\rho_n^Y \circ \Sigma\alpha_n$ whose equivalence class is equal in $\pi_k(X)$ to that of α_n. Therefore, $f_*(\beta) = \alpha$. □

Using the above and a bit more work, one can prove the following (compare it to Corollary 2.29):

Proposition 5.13. *There are natural weak equivalences $\Sigma Pf \to Cf$ and $Pf \to \Omega Cf$. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of spectra is a fiber sequence if and only if it is a cofiber sequence.*

PROOF. Omitted. □

As a direct consequence of the long exact sequence of homotopy groups for fiber/cofiber sequences, we have:

Corollary 5.14. *Let $f : X \to Y$ be a map of spectra. Then the following are equivalent:*

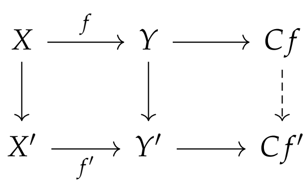
(1) f is an equivalence,
(2) The homotopy fiber of f is weakly equivalent to the zero spectrum,
(3) The homotopy cofiber of f is weakly equivalent to the zero spectrum,

Thus, the analogy that "(co)fibers are like (co)kernels" doesn't work too well. We should rather be thinking of cofiber sequences as triangles... more on that below. To illustrate that better, in Section 3 we will give an example of maps which in classical algebra are injective or surjective, but when transported to the world of spectra, they have interesting fibers and cofibers.

3. Homotopy pushouts and pullbacks

We could define the *double mapping cylinder* in spectra, and then we'd say a square is *homotopy cocartesian* if the induced map from the double mapping cylinder is a weak equivalence. We could then prove it is equivalent to the following, which for simplicity we choose to simply give as the definition.

Definition 5.15. The commutative square in the left of the following diagram is a *homotopy pushout* if the induced dotted arrow is a weak equivalence.



Similarly, define *homotopy pullbacks*.

The following shows that the Blakers–Massey Theorem 2.27 is much simplified in the realm of spectra.

Corollary 5.16. *A commutative square of spectra is a homotopy pullback if and only if it is a homotopy pushout.*

PROOF. We have the following commutative diagram

$$\begin{array}{ccccccccccc} \Omega Cf & \xleftarrow{\sim} & Pf & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Cf & \xleftarrow{\sim} & \Sigma Pf \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega Cf' & \xleftarrow[\sim]{} & Pf' & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Cf' & \xleftarrow[\sim]{} & \Sigma Pf' \end{array}$$

The proof follows from 2-out-of-3 and Proposition 5.1. □

This is one of the main properties of Ho(Sp). It is essentially equivalent to stability, in the sense of the equivalence in Proposition 5.13. See [Lur17, 1.1.3.4] for a precise statement.

4. Product and coproduct

In the category of abelian groups, or in the category of R-modules for any commutative ring R, finite products and coproducts coincide. This is true in any abelian category. It is also true in the homotopy category of spectra, as we shall now see. It is not true in spaces, and a precise measure of how it fails to be true was given in Corollary 2.30.

First, note that in any category with binary (co)products and a zero object, there is a map from the binary coproduct to the binary product, by the universal properties, using the zero maps $X \to Y$ and $Y \to X$.

$$\begin{array}{ccccc} X & & & & X \\ & \searrow^{\iota_0} & & \nearrow^{\pi_0} & \\ & & X \vee Y \xrightarrow{c} X \times Y & & \\ & \nearrow_{\iota_1} & & \searrow_{\pi_1} & \\ Y & & & & Y \end{array} \tag{5.17}$$

Lemma 5.18. *Let $X, Y \in \mathrm{Sp}$. The canonical map $\pi_k(X) \oplus \pi_k(Y) \xrightarrow{(\iota_{0*}, \iota_{1*})} \pi_k(X \vee Y)$ is an isomorphism.*[7]

PROOF. Since by construction $X \xrightarrow{\iota_0} X \vee Y \xrightarrow{c} X \times Y \xrightarrow{\pi_1} Y$ is the zero map, we get an induced map $C\iota_0 \to Y$, and this map is a level equivalence, therefore a weak equivalence, so we have a cofiber sequence

$$X \xrightarrow{\iota_0} X \vee Y \xrightarrow{\pi_1 c} Y.$$

It is split on the right by $\iota_1 : Y \to X \vee Y$, so by Proposition 5.12 we have a split exact sequence

$$\pi_{k+1}(X \vee Y) \underset{(\iota_1)_*}{\rightleftarrows} \pi_{k+1}(Y) \xrightarrow{\partial} \pi_k(X) \longrightarrow \pi_k(X \vee Y) \underset{(\iota_1)_*}{\rightleftarrows} \pi_k(Y) \xrightarrow{\partial} \pi_{k-1}(X)$$

which implies that

$$0 \longrightarrow \pi_k(X) \longrightarrow \pi_k(X \vee Y) \underset{(\iota_1)_*}{\rightleftarrows} \pi_k(Y) \longrightarrow 0$$

is a split short exact sequence, proving the result. □

Corollary 5.19. *If $f : X \to Y$ and $g : X' \to Y'$ are weak equivalences, then $f \vee g : X \vee X' \to Y \vee Y'$ is a weak equivalence.*

[7] It follows by induction that the analogous result is true for a finite number of spectra. It's also true for arbitrary wedges.

Corollary 5.20. *Let X and Y be spectra. The map $c : X \vee Y \to X \times Y$ is an equivalence.*

PROOF. Taking homotopy groups in (5.17), we get a sequence

$$\pi_k(X) \oplus \pi_k(Y) \longrightarrow \pi_k(X \vee Y) \longrightarrow \pi_k(X \times Y) \longrightarrow \pi_k(X) \times \pi_k(Y).$$

The first map is an isomorphism by the previous lemma. The last map is an isomorphism by Exercise 4.12. The total composite is an isomorphism by construction. Therefore, the middle map is an isomorphism. □

The corollary above gets us close to proving what we want, but we first need to see that finite products and coproducts in $\mathrm{Ho}(\mathrm{Sp})$ are really given by $\vee$ and $\times$ from Sp.

Proposition 5.21. *The binary coproduct in $\mathrm{Ho}(\mathrm{Sp})$ is given by $\vee$, and the binary product by $\times$. The natural map $X \vee Y \to X \times Y$ in $\mathrm{Ho}(\mathrm{Sp})$ is an isomorphism; this is summarized by saying that $\mathrm{Ho}(\mathrm{Sp})$ is* semiadditive*.*

PROOF. Let X, Y, Z be spectra. Since the coproduct of cell spectra is cell, $QX \vee QY$ is cell. Also, $QX \vee QY \to X \vee Y$ is a weak equivalence by Corollary 5.19. What we want to prove is that the top map in the following commutative diagram is an isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Ho(Sp)}}(X \vee Y, Z) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Z) \times \mathrm{Hom}_{\mathrm{Ho(Sp)}}(Y, Z) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_{\mathrm{Ho(Sp)}}(QX \vee QY, RZ) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ho(Sp)}}(QX, RZ) \times \mathrm{Hom}_{\mathrm{Ho(Sp)}}(QY, RZ) \end{array}$$

So it suffices to see the bottom map is an isomorphism. But all the domains in the hom-sets are cell, and all the domains are Ω-spectra, so by Exercise 4.57, we want to see that

$$[QX \vee QY, RZ] \to [QX, RZ] \times [QY, RZ]$$

is an isomorphism. This follows from the universal property for $\vee$ in Sp, plus the fact that homotopies out of a wedge are in bijection with pairs of homotopies from each of the factors, because the smash product of pointed spaces commutes with the wedge.

The final claim follows from Corollary 5.20. □

Remark 5.22. Observe that the zero spectrum is a cell Ω-spectrum, and it is the zero object in $\mathrm{Ho}(\mathrm{Sp})$. By induction, we deduce that $\mathrm{Ho}(\mathrm{Sp})$ has finite products and coproducts, and that the natural map $X_1 \vee \cdots \vee X_n \to X_1 \times \cdots \times X_n$ is an isomorphism. [8]

The above can be generalized a bit: if you have a set $\{X_\alpha\}$ of spectra such that for each X_α, all but a finite number of the $\pi_n(X_\alpha)$ are zero, then the natural map $\bigvee_\alpha X_\alpha \to \prod_\alpha X_\alpha$ is an equivalence. This is [**Ada74**, III.3.14].

As with $\mathrm{Ho}(\mathrm{Top})$, that's generally it for strict colimits and limits in $\mathrm{Ho}(\mathrm{Sp})$. The important notion here is that of general *homotopy* limits and colimits, but we shall not get into that.

[8] $\mathrm{Ho}(\mathrm{Sp})$ actually also has arbitrary coproducts and they coincide with the ones of Sp, see Footnote 7. The analogous statement for infinite products is false, see [**Sch**, I.2.20] for an explicit counterexample.

CHAPTER 6

More structure

1. Additivity

We just proved that $\mathrm{Ho}(\mathrm{Sp})$ is a semiadditive category, i.e. it has a zero object and a *biproduct* $A \oplus B$ (which is both the product and the coproduct in a compatible way). Then, as in any semiadditive category, we can define a sum operation on homsets like this: if $f, g : A \to B$, define $f + g : A \to B$ to be the composition

$$A \xrightarrow{\Delta} A \times A = A \oplus A \xrightarrow{f \oplus g} B \oplus B = B \vee B \xrightarrow{\nabla} B.$$

This endows $\mathrm{Hom}(A, B)$ with the structure of a commutative monoid, and composition is bilinear, so that $\mathrm{Ho}(\mathrm{Sp})$ is enriched over the category of commutative monoids.[1] Now, let

$$A \times A \cong A \vee A \to A \times A$$

be (ι_0, Δ): this is the *shearing map* $(a, a') \mapsto (a + a', a')$. It is a general remark that it is an isomorphism if and only if $\mathrm{Map}(A, X)$ has inverses and is thus an abelian group, for all X. In this case, composition is also a morphism of abelian groups, and we say the category is *additive*. See [**Sch**, II.1.12] for details.

Proposition 6.1. *The shearing map is an isomorphism for all spectra A, and thus* $\mathrm{Ho}(\mathrm{Sp})$ *is an additive category.*

Proof. We want to prove that $(\iota_0, \Delta) : A \vee A \to A \times A$ is a weak equivalence. To do that, consider the following diagram:

$$\begin{array}{ccc} \pi_k(A \vee A) & \xrightarrow{(\iota_0,\Delta)_*} & \pi_k(A \times A) \\ {\scriptstyle (\iota_{0*},\iota_{1*})}\uparrow\cong & & \cong\downarrow{\scriptstyle (\pi_{0*},\pi_{1*})} \\ \pi_k(A) \oplus \pi_k(A) & \longrightarrow & \pi_k(A) \times \pi_k(A) \end{array}$$

The bottom map is the shearing map of $\pi_k(A)$ in the category of abelian groups. It makes the diagram commute by inspection, and it is an isomorphism because $\pi_k(A)$ is an abelian group. Therefore, the top map is an isomorphism, for all $k \in \mathbb{Z}$. □

Remark 6.2.

(1) Thus, additivity of a category is *property*, not *structure*: it is the property of there existing a zero object, of the canonical map from the binary coproduct to the binary product being an isomorphism, and of the shear map being an isomorphism. The structure of enrichment over abelian groups comes for free. Some definitions of an additive category are of the form "semiadditive + enrichment over abelian groups" – you could then check that the two sums have to be the same.

(2) We can give a more explicit enrichment of $\mathrm{Ho}(\mathrm{Sp})$ over Ab, which has to coincide with the one we gave by the general property of the previous item. The idea is to exploit Proposition 2.11 and the fact that Σ is an equivalence of categories in $\mathrm{Ho}(\mathrm{Sp})$. First, note that Proposition 2.11 generalizes *mutatis mutandis* to other categories. Now, in

[1] The monoidal product in the category of commutative monoids is given by the tensor product of commutative monoids. You may not be familiar with it, but it's very much analogous to the tensor product of abelian groups.

Ho(Sp), a double suspension is a homotopy cocommutative co-H-group object, similarly as in spaces. Since $\Sigma^2 : \mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y) \to \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^2 X, \Sigma^2 Y)$ is a bijection, then $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y)$ acquires an abelian group structure, and you can prove this defines an enrichment of Ho(Sp) over Ab. You could do something analogous using Ω^2, and it would give the same structure, too.

2. Some more results on homsets, and graded homotopy groups

Let us see how the abelian group structures above generalize those of the homotopy groups. To this end, it is comfortable to see Ho(Sp) not only as enriched over Ab, but as being enriched over $\mathrm{Ab}^{\mathbb{Z}}$, i.e. as having graded abelian groups as hom-objects:

Definition 6.3. For $X, Y \in \mathrm{Sp}$ and $n \in \mathbb{Z}$, define

$$\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y)_n := \mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, \Sigma^{-n} Y) \cong \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^n X, Y).$$

Define the composition

$$\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y)_n \times \mathrm{Hom}_{\mathrm{Ho(Sp)}}(Y, Z)_m \to \mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Z)_{n+m}$$

as follows: if $f : X \to \Sigma^{-n} Y$ and $g : Y \to \Sigma^{-m} Z$, then its composition is

$$X \xrightarrow{f} \Sigma^{-n} Y \xrightarrow{\Sigma^{-n} g} \Sigma^{-n-m} Z.$$

Thus, $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y)_\bullet$ is a $\mathbb{Z}$-graded abelian group, and composition is a morphism of graded abelian groups.

Our aim is to prove that $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\mathbb{S}, X)_\bullet \cong \pi_*(X)$, which justifies the choice of the sign in the grading. To do this, we will first prove some useful observations about maps.

Proposition 6.4. *Let A be a pointed cell complex, Y be an Ω-spectrum and $d \geq 0$. Then*

$$\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{\infty-d} A, Y) \cong [A, Y_d].$$

PROOF. By adjunction, we have $\mathrm{Hom}_{\mathrm{Top}_*}(A, Y_d) \cong \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^{\infty-d} A, Y)$, and you can check that this is compatible with homotopies so that $[A, Y_d] \cong [\Sigma^{\infty-d} A, Y]$. Now, since A is a cell complex, then $\Sigma^{\infty-d} A$ is a cell spectrum, so since Y is an Ω-spectrum by hypothesis, then this is really isomorphic to $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{\infty-d} A, Y)$. □

Proposition 6.5. *Let A be a pointed finite cell complex, X be any spectrum and $d \geq 0$. Then*

$$\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{\infty-d} A, X) \cong \mathrm{colim}_n [\Sigma^n A, X_{d+n}].$$

In particular, if A, B are pointed cell complexes and A is finite, then

$$\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^\infty A, \Sigma^\infty B) \cong \mathrm{colim}_n [\Sigma^n A, \Sigma^n B], \tag{6.6}$$

and similarly for $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{\infty-d} A, \Sigma^{\infty-e} B)$.

PROOF. Since X is not an Ω-spectrum, we need to replace it by RX in order to compute this. Now, recall the definition of RX and the results from Section 1.5: remember that a finite cell complex is compact. Using the proposition above, we get:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{\infty-d} A, X) &\cong [A, (RX)_d] = [A, \mathrm{hocolim}_n \Omega^n X_{d+n}] \cong \\ &\cong \mathrm{colim}_n [A, \Omega^n X_{d+n}] \cong \mathrm{colim}_n [\Sigma^n A, X_{d+n}].\end{aligned}$$ □

Remark 6.7. Above, we cannot dispense with the hypothesis of A being finite, in general. The failure of that isomorphism is measured by a certain $\lim^1$ term fitting in *Milnor's exact sequence*. See e.g. [**Sch**, II.5.8].

Look at the isomorphism (6.6): the right-hand side makes no reference to spectra, and is the group $\{X, Y\}$ we defined back in Definition 2.24.1. We see how the study of stable maps between spaces is a particular case of the more general study of maps between spectra.

Just as homotopy groups of pointed spaces are given by $[S^k, X]$ for $k \geq 0$, the same is true for spectra for $k \in \mathbb{Z}$, as soon as you define $\mathbb{S}^k$ as $\Sigma^k\mathbb{S}$, or, a bit more explicitly, if $k \geq 0$ then $\mathbb{S}^k$ is $\Sigma^\infty S^k$ and $\mathbb{S}^{-k}$ is $\Sigma^{\infty - k} S^0$.

Corollary 6.8. *Let X be a spectrum. Then* $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\mathbb{S}, X)_\bullet \cong \pi_*(X)$.

PROOF. If $k \geq 0$, then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\mathbb{S}, X)_k = \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^k\mathbb{S}, X) \cong \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^\infty S^k, X) \cong \\ \cong \mathrm{colim}_n [S^{n+k}, X_n] = \pi_k(X).\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\mathbb{S}, X)_{-k} = \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{-k}\mathbb{S}, X) \cong \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{\infty-k} S^0, X) \cong \\ \cong \mathrm{colim}_n [S^{n}, X_{k+n}] = \pi_{-k}(X).\end{aligned}$$

□

Example 6.9. $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\mathbb{S}, \mathbb{S}) \cong \pi_0(\mathbb{S}) \cong \mathbb{Z}$.

3. Triangulated structure

We will only give a little sketch here. A *triangulated category* is an additive category $\mathcal{C}$ endowed with an endofunctor $\Sigma : \mathcal{C} \to \mathcal{C}$ which is an equivalence of categories, and endowed with a class of sequences, called *distinguished triangles*, of the form $X \to Y \to Z \to \Sigma X$, satisfying some axioms. One can prove that the category $\mathrm{Ho(Sp)}$ is a triangulated category, though we shan't do it. The endofunctor is the suspension functor, and the distinguished triangles are the ones isomorphic to images under γ of cofiber sequences: i.e. there exists a map $u : A \to B$ in Sp with homotopy cofiber $i : B \to Cu$, and connecting map $\delta : Cu \to \Sigma A$ (that the homotopy cofiber of i is of this form was observed in passing in Section 2, by analogy to the case of pointed spaces), fitting into a commutative diagram in $\mathrm{Ho(Sp)}$

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \cong\downarrow a & & \downarrow\cong & & \downarrow\cong & & \cong\downarrow\Sigma a \\ A & \xrightarrow[\gamma(u)]{} & B & \xrightarrow[\gamma(i)]{} & Cu & \xrightarrow[\gamma(\delta)]{} & \Sigma A. \end{array}$$

Some of the axioms are: any map $X \to Y$ can be completed to a distinguished triangle; $* \to X \xrightarrow{\mathrm{id}} X \to *$ is a triangle; you can continue forming distinguished triangles as in the Puppe sequence: if (f, g, h) is a triangle, then $(g, h, -\Sigma f)$ is a triangle, and the converse is also true; if you are given a ladder diagram of triangles without a map in the third column and with the suspension of the first map in the fourth column, then you can fill in the third column. There is an extra axiom called the *octahedral axiom* which is a bit more complicated.

For example, to see that every map can be completed to a distinguished triangle, we use the construction from Lemma 4.52.(3):

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\gamma(i)\circ b} & Cf' & \xrightarrow{(\Sigma a)^{-1}\circ\gamma(\delta)} & \Sigma X \\ {\scriptstyle a}\downarrow\cong & & {\scriptstyle b}\downarrow\cong & & {\scriptstyle \mathrm{id}}\downarrow\cong & & \cong\downarrow{\scriptstyle \Sigma a} \\ QRX & \xrightarrow[\gamma(f')]{} & QRY & \xrightarrow[\gamma(i)]{} & Cf' & \xrightarrow[\gamma(\delta)]{} & \Sigma QRX \end{array}$$

From the axioms it follows that you can also extend distinguished triangles to the left: if (f, g, h) is a distinguished triangle, then $(-\Omega h, f, g)$ is a distinguished triangle. Indeed, just see h as $-\Sigma(-\Sigma^{-1}h) : \Sigma\Sigma^{-1}Z \to \Sigma\Sigma^{-1}\Sigma X$.

Knowing that $\mathrm{Ho(Sp)}$ has this triangulated structure, we can exploit the theory of triangulated categories (see e.g. [Nee01]) to deduce some theorems. For example [Sch, II.2.10]:

Proposition 6.10. *Let $X \xrightarrow{f} Y \xrightarrow{g} C$ be a fiber/cofiber sequence in $\mathrm{Ho(Sp)}$. Then, for every spectrum W, there are long exact sequences of abelian groups*

$$\cdots \longrightarrow \mathrm{Hom}_{\mathrm{Ho(Sp)}}(W,\Omega Y) \xrightarrow{-(\Omega g)_*} \mathrm{Hom}_{\mathrm{Ho(Sp)}}(W,\Omega C) \longrightarrow \mathrm{Hom}_{\mathrm{Ho(Sp)}}(W,X) \xrightarrow{f_*} \mathrm{Hom}_{\mathrm{Ho(Sp)}}(W,Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathrm{Ho(Sp)}}(W,C) \longrightarrow \mathrm{Hom}_{\mathrm{Ho(Sp)}}(W,\Sigma X) \xrightarrow{-(\Sigma f)_*} \mathrm{Hom}_{\mathrm{Ho(Sp)}}(W,\Sigma Y) \longrightarrow \cdots$$

Similarly, but displayed more succinctly for reading comfort,

$$\cdots \longrightarrow \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma X, W) \longrightarrow \mathrm{Hom}_{\mathrm{Ho(Sp)}}(C, W) \xrightarrow{g^*} \mathrm{Hom}_{\mathrm{Ho(Sp)}}(Y, W) \xrightarrow{f^*} \mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, W) \longrightarrow \cdots$$

Remark 6.11. This is a follow-up to Remark 4.39. In general, to define homotopy colimits, you cannot do it merely in the homotopy category: you have forgotten data that's necessary to build them. In the case of sequential homotopy colimits of spectra, however, it can be done by using a clever trick. See [Sch, II.5.3] for the definition of the homotopy colimit of a sequence in a triangulated category. Using this, one can prove what was announced in that previous remark: $X \simeq \mathrm{hocolim}_n \Sigma^{\infty-n} X_n$ for any spectrum X. Schwede proves it along these lines in [Sch, II.5.12] for symmetric spectra.

Side remark 6.12. You could ask yourself the question: does $\mathrm{Ho(Sp)}$ together with its triangulated structure completely describe the homotopy theory of spectra? That is a vague question, but it can be made precise. You need a stronger hypothesis to get a positive answer: you need to know that the triangulated structure comes from stability before taking the homotopy category. More precisely: Sp is not merely a model category, it is a *stable* model category, which means essentially that it has a canonical suspension functor which becomes an equivalence once you pass to the homotopy category. This is enough for $\mathrm{Ho(Sp)}$ to get a triangulated structure [Hov99, Section 7].[2] Now, suppose that $\mathcal{C}$ is another stable model category, and suppose that $\mathrm{Ho}(\mathcal{C}) \simeq \mathrm{Ho(Sp)}$ as triangulated categories. Does this mean that the homotopy theory of Sp and that of $\mathcal{C}$ are equivalent? More precisely, does it mean that $\mathcal{C}$ and Sp are Quillen-equivalent? Yes. This is a theorem of Schwede [Sch07]. Even more precisely, a triangulated equivalence $\Phi : \mathrm{Ho(Sp)} \to \mathrm{Ho}(\mathcal{C})$ can be lifted to a Quillen equivalence $\mathrm{Sp} \to \mathcal{C}$ which takes $\mathbb{S}$ to a cofibrant-fibrant object which is isomorphic to $\Phi(\mathbb{S})$.

Here's another question you may ask. Is Sp special in some way, among all the stable model categories? Yes, it is. It is the "free stable model category on one object". This means that if $\mathcal{C}$ is another stable model category and $X \in \mathcal{C}$ is a cofibrant-fibrant object, then you get a Quillen left

[2] Without the very reasonable extra hypothesis of being induced from a stable model category, a related conjecture of Margolis (which adds some other axioms), already alluded to in Remark 4.58, is open as per MO:Is Margolis's axiomatisation conjecture still alive?.

adjoint $X \otimes - : \mathrm{Sp} \to \mathcal{C}$ that takes $\mathbb{S}$ to X. This is a theorem of Schwede and Shipley [SS02, 5.1] which was used by Schwede in the proof of the result at the end of the previous paragraph.[3]

4. The smash product

There is a monoidal product in spectra, akin to the smash product of pointed spaces, or to the tensor product of abelian groups or of their chain complexes. It is not that easy to construct, as we mentioned in the introduction. Ideally, we would like there to be a functor $\wedge$ defined in Sp which endows Sp with the structure of a symmetric monoidal category, and which is suitably compatible with the homotopical structure, so that $\mathrm{Ho}(\mathrm{Sp})$ gets an induced symmetric monoidal structure. Well, we cannot do this in our category Sp. You could try constructing it, but then you don't end up actually satisfying the axioms. You can maneuver around this and get merely a symmetric monoidal structure in $\mathrm{Ho}(\mathrm{Sp})$. If you're really interested, you could read about it in [Ada74, III.4] or [Swi75, Page 254ff.], but it's probably not worth it to work out the details.

We can hastily say the following: the idea, to quote Adams, is that $X \wedge Y$ is "the thing to which $X_n \wedge Y_m$ tends as n, m tend to infinity". The complication comes from the structure maps; one problem comes from the arbitrariness of having to choose when to apply the structure map on the left and when on the right; another problem stems from, once again, the fact that the twist $S^1 \wedge S^1 \to S^1 \wedge S^1$ has degree -1. An interesting exposition of the problems one runs into is in [Rog17, 3.8]. One possible (among many) explicit definition of such a "handicrafted smash product", as they're called, is the following. For $X, Y \in \mathrm{Sp}$, define

$$(X \wedge Y)_k = \begin{cases} X_n \wedge Y_n & \text{if } k = 2n \\ X_{n+1} \wedge Y_n & \text{if } k = 2n+1. \end{cases}$$

For the structure maps, use

$$\Sigma(X_n \wedge Y_n) \cong \Sigma X_n \wedge Y_n \xrightarrow{\rho_n^X \wedge \mathrm{id}} X_{n+1} \wedge Y_n$$

and

$$\Sigma(X_{n+1} \wedge Y_n) = S^1 \wedge X_{n+1} \wedge Y_n \xrightarrow{\tau \wedge \mathrm{id}} X_{n+1} \wedge S^1 \wedge Y_n \xrightarrow{\mathrm{id} \wedge \rho_n^Y} X_{n+1} \wedge Y_{n+1}.$$

We shall not be using this construction (or other constructions) at all. A reader familiar with the tensor product of chain complexes might suggest setting $(X \wedge Y)_n = \bigvee_{p+q=n} X_p \wedge Y_q$, but there is no sensible way to define the structure maps.

The more modern model categories of spectra, such as the already mentioned symmetric spectra [HSS00], [Sch], orthogonal spectra [MMSS01] or EKMM spectra [EKMM97] solve this problem: their homotopy categories are equivalent to $\mathrm{Ho}(\mathrm{Sp})$ but they have the advantage of having a monoidal structure before passing to homotopy. In the case of symmetric and orthogonal spectra, the smash product is defined very categorically, and is an application of the Day convolution product. Lurie's approach to stable homotopy theory using ∞-categories [Lur17] also solves this problem.

Side remark 6.13. As a historical remark: the quest for a good monoidal model of spectra took many years, many different attempts, and many different workarounds. When Lewis published the short article [Lew91], it was a bad blow to the hopes of success: he proved that there could be no symmetric monoidal category of spectra satisfying a short list of reasonable axioms. Luckily, one can drop a single one of them and eliminate the problem, which was first done by [EKMM97] and then by [HSS00]; if memory serves right, they both drop a different

[3] In both of these papers, Sp is a model for spectra very close to ours, where spaces are replaced with simplicial sets.

axiom. See also MO:Is the ∞-category of spectra convenient? for interesting discussion. A discussion of this problem in the context of EKMM spectra is in [Elm04].

We now reproduce the motivation from [Rog17, 3.8] because it is pleasantly down-to-earth. Let R be a ring. Then $\widetilde{H}^*(-;R)$ gets a graded product (the cup product), induced from an external product

$$\widetilde{H}^n(X;R) \otimes \widetilde{H}^n(Y;R) \to \widetilde{H}^{n+m}(X \wedge Y;R) \tag{6.14}$$

for every $X, Y \in \mathrm{Top}_*$. Now, we know that $\widetilde{H}^n(X;R) \cong [X, K(R,n)]$, and recall that the $K(R,n)$ assemble to an Ω-spectrum HR. The above external product can be seen to be induced by a graded product in the Eilenberg–Mac Lane spaces, i.e. there are maps

$$\phi_{n,m} : K(R,n) \wedge K(R,m) \to K(R,n+m)$$

such that (6.14) takes $[X \xrightarrow{f} K(R,n)] \otimes [Y \xrightarrow{g} K(R,m)]$ to the composition

$$X \wedge Y \xrightarrow{f \wedge g} K(R,n) \wedge K(R,m) \xrightarrow{\phi_{n,m}} K(R,n+m) \ .$$

An explicit construction of the $\phi_{n,m}$ in a more general setting is in [Sto18, 3.2.1]. The fact that it recovers the usual product was probably first proven in [TW80].

The question we now ask ourselves is: since we can represent $\widetilde{H}^*(-;R)$ by a spectrum HR with $(HR)_n = K(R,n)$, can we also represent these products at the spectra level? More precisely, does there exists a spectrum $HR \wedge HR$ with maps $i_{n,m} : (HR)_n \wedge (HR)_m \to (HR \wedge HR)_{n+m}$ and a spectrum map $\mu : HR \wedge HR \to HR$ such that the $\phi_{n,m}$ factor as follows?

$$\begin{array}{ccc} (HR)_n \wedge (HR)_m & \xrightarrow{\quad\phi_{n,m}\quad} & (HR)_{n+m} \\ & {\scriptstyle i_{n,m}} \searrow \qquad \nearrow {\scriptstyle \mu_{n+m}} & \\ & (HR \wedge HR)_{n+m} & \end{array}$$

As above, the answer is yes, but the operation $\wedge$ doesn't behave well in our category Sp, merely in Ho(Sp). We state the following without proof.

Proposition 6.15. *There exists a functor* $- \wedge - : \mathrm{Ho(Sp)} \times \mathrm{Ho(Sp)} \to \mathrm{Ho(Sp)}$ *called the* smash product *such that:*

1. Ho(Sp) *is a symmetric monoidal category, with monoidal product* $- \wedge -$ *and unit* $\mathbb{S}$.
2. *It is closed, i.e.* $X \wedge -$ *has a certain right adjoint* $F(X,-)$*, the internal* function spectrum.
3. *This structure is compatible with the operations over* Top_**: if* A *is a cell complex and* X *is a spectrum, then* $X \wedge A \cong X \wedge \Sigma^\infty A$ *and* $F(A,X) \cong F(\Sigma^\infty A, X)$*.*
4. *Smashing a cofiber sequence of spectra with a given spectrum gives a cofiber sequence of spectra.*
5. $\Sigma^\infty(A \wedge B) \cong \Sigma^\infty A \wedge \Sigma^\infty B$ *for cell complexes* A *and* B*.*
6. $\Sigma(X \wedge Y) \cong \Sigma X \wedge Y \cong X \wedge \Sigma Y$*.*
7. *The functor* $H : \mathrm{Ab} \to \mathrm{Ho(Sp)}$ *is symmetric monoidal, so* $H(A \otimes B) \cong HA \wedge HB$*.*

Here $\otimes$ means $\otimes_{\mathbb{Z}}$, and similarly $\wedge$ should be thought of as $\wedge_{\mathbb{S}}$.

Remark 6.16. You may be wondering how are the homotopy groups $\pi_*(X \wedge Y)$ related to the homotopy groups $\pi_*(X)$ and $\pi_*(Y)$. Without further hypothesis, the best answer is: there is a spectral sequence [EKMM97, IV.1.1, IV.4.1]

$$E^2_{p,q} = \mathrm{Tor}^{\pi_*\mathbb{S}}_{p,q}(\pi_* X, \pi_* Y) \Rightarrow \pi_{p+q}(X \wedge Y)$$

where $\pi_*\mathbb{S}$ has a graded ring structure presented in the exercises.

With additional connectivity hypotheses on X and Y, this spectral sequence degenerates. If X is $(n-1)$-connected and Y is $(m-1)$-connected, then $X \wedge Y$ is $(n+m-1)$-connected and

$\pi_n(X) \otimes \pi_m(Y) \cong \pi_{n+m}(X \wedge Y)$, see [Sch, II.5.22]. Thus, for example, the smash product of connective spectra is connective, and its π_0 is the tensor product of the π_0's.

Side remark 6.17. In Side remark 6.12, we observed how special Sp is among stable model categories. If we take the smash product into consideration, we can say more. Let Sp^Σ denote the symmetric monoidal model category of symmetric spectra with the smash product. Then a model category $\mathcal{C}$ is stable if and only if it is tensored, enriched and cotensored over Sp^Σ. I may be missing a couple of technical hypotheses here. The original, model-categorical theorem is due to Shipley [Shi01] but the formulation is a bit different; an ∞-categorical formulation along these lines can be found in [Lur17, 4.8.2.10].

A very high-brow way of saying this is the following; beware, what follows is way more technological than what we have at hand, now, but maybe you'll find it interesting. Lurie considers the ∞-category Pr^L of presentable ∞-categories with left adjoint functors as morphisms. He endows it with the structure of a symmetric monoidal ∞-category, and this structure is such that commutative monoids therein are precisely presentable closed symmetric monoidal ∞-categories. One such object is the ∞-category Sp of spectra with the smash product. It is stable, and indeed Sp plays a very special role among these: the ∞-category of stable presentable ∞-categories is equivalent to the ∞-category of Sp-modules in Pr^L. See also [GGN15].

CHAPTER 7

Further properties and examples

1. (Co)homology theories, II

We are now ready to resume our work on cohomology theories.

In Proposition 3.9 we saw that if E is an Ω-spectrum[1], then the functors $E^n(-) : CW_*^{\mathrm{op}} \to \mathrm{Ab}$ given by

$$E^n(-) = \begin{cases} [-, E_n] & \text{if } n \geq 0 \\ [\Sigma^{-n}(-), E_0] & \text{if } n < 0 \end{cases}$$

define an extraordinary cohomology theory on pointed CW-complexes. Let us extend this definition to general spectra.

Definition 7.1. Let X be a pointed CW complex and E be a spectrum. For $n \in \mathbb{Z}$, define abelian groups

$$E^n(X) := \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^\infty X, E)_{-n} = \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{-n}\Sigma^\infty X, E),$$

where if $n \geq 0$, then this is isomorphic to $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{\infty-n} X, E)$ by Remark 5.3.

Remark 7.2. (1) This really recovers the previous definition where E was an Ω-spectrum, by adjunction when $n < 0$ and by Proposition 6.4 when $n \geq 0$.

(2) We have

$$E^n(X) \cong \pi_{-n} F(X, E) \cong \mathrm{colim}_k \pi_{-n+k} F(X, E_k). \tag{7.3}$$

Indeed, by Corollary 6.8 and Remark 5.3, we have

$$\pi_{-n}F(X,E) \cong \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{-n}\mathbb{S}, F(X,E)) \cong \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{-n}\mathbb{S}\wedge X, E) \cong \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{\infty-n}X, E).$$

Proposition 7.4. *For any spectrum E, the functors $E^n(-)$ form an extraordinary, reduced cohomology theory on pointed CW-complexes.*

PROOF. The suspension natural isomorphism is gotten as:

$$\begin{aligned} E^{n+1}(\Sigma X) &= \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{-n-1}\Sigma^\infty \Sigma X, E) \\ &\cong \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{-1}\Sigma^{-n}\Sigma^\infty X, \Sigma^{-1}E) \xleftarrow[\sim]{\Sigma^{-1}} \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^{-n}\Sigma^\infty X, E) = E^n(X). \end{aligned}$$

Combining Proposition 6.10 and Example 5.10, we immediately see that E-cohomology satisfies exactness. As for additivity, it follows from the fact that wedges are the coproducts in Ho(Sp) (see Footnote 8), so the contravariant Ab-enriched hom takes them to direct products in Ab. □

Example 7.5. Take $E = \mathbb{S}$. Then $E^*(-)$ is *stable cohomotopy theory*. Notice what happens unstably: if X is a pointed space, then $[X, S^n]$ has no reason to be an abelian group, it's just a pointed set.

What about homology theories? Yes, we can also do that. Define a *homology theory* on pointed CW-complexes dually to Definition 3.4: the functors are covariant.

A spectrum gives rise to a homology theory, dualizing description (7.3) as follows:

[1] Somehow, it is common in the literature to use E to denote a spectrum that we intend to see as a (co)homology theory.

Definition 7.6. Let E be a spectrum. For $n \in \mathbb{Z}$, define $E_n(-) : CW_*^{\mathrm{op}} \to \mathrm{Ab}$ by

$$E_n(X) = \pi_n(E \wedge X) \cong \mathrm{colim}_k \pi_{n+k}(E_k \wedge X).$$

The proof of the following proposition is dual to that of Proposition 7.4.

Proposition 7.7. *For any spectrum E, the functors $E_n(-)$ form an extraordinary, reduced homology theory on pointed CW-complexes.*

There is a possible clash of notations between the abelian group $E_n(X)$ and the pointed space E_n, but it should be clear from context what do we mean.

Example 7.8.

(1) Take $E = \mathbb{S}$. Then $E_n(-)$ is stable homotopy theory, i.e. $E_n(X) = \pi_n^s(X)$. This is an abstract proof of π_*^s forming a homology theory: you first prove that any spectrum gives rise to a homology theory, and then you remark that the sphere spectrum gives rise to the stable homotopy groups. You can read a direct proof in [**Hat02**, 4F.1].

(2) If $E = KU$, then KU-homology is interesting, but we shall not enter into that here. See e.g. MO:Why the Dold-Thom theorem? for a description and a pointer to further reading.

(3) Take $E = HG$ for an abelian group G. Then $HG_n(S^0)$ is G if $n = 0$ and 0 if not, so $HG_*(-) \cong \widetilde{H}_*(-;G)$ by uniqueness of ordinary reduced homology theories.

(4) Let E be a spectrum. Then $(\Sigma^k E)_*(-) \cong E_{*+k}(-)$. In particular,

$$(\Sigma^k HG)_*(-) \cong \widetilde{H}_{*+k}(-;G).$$

(5) We can introduce *generalized* or *graded Eilenberg–Mac Lane spectra*. If $G_\bullet$ is a $\mathbb{Z}$-graded abelian group, then define $HG_\bullet := \bigvee_n \Sigma^n HG_n \simeq \prod_n \Sigma^n HG_n$, where the equivalence follows from Remark 5.22. We deduce that

$$(HG_\bullet)_q(X) \cong \sum_n \widetilde{H}_{q-n}(X;G_n) \quad \text{and} \quad (HG_\bullet)^q(X) \cong \prod_n \widetilde{H}^{q-n}(X;G_n).$$

You can extend H to a functor $\mathrm{GrAb}_\mathbb{Z} \to \mathrm{Ho}(\mathrm{Sp})$.

Remark 7.9. Remember Grothendieck's wisdom: only take homology groups if you can't escape it; it's the object before taking homology that contains the whole information. The same rule of thumb applies here, with homotopy instead. So we can say that the spectrum $F(X,E)$ is, really, the spectrum of cohomology of X with coefficients in E, and $E \wedge X$ is, really, the spectrum of homology of X with coefficients in E.

Remark 7.10. Remember how ordinary homology can be obtained as the homology of a certain chain complex? You start with an abelian group G, you build a certain singular chain complex of abelian groups $C_*(X;G)$, and then its homology is $H_*(X;G)$. It is a theorem of Burdick–Conner–Floyd [**BCF68**] that ordinary homology and its sums, as in Example 7.8.(5), are the only homology theories that you can get via chain complexes of abelian groups like this.

Definition 7.11. Let E be a spectrum. The *coefficient groups* of the homology theory E_* are given by

$$E_*(S^0) = \pi_*(E \wedge S^0) \cong \pi_*(E),$$

and those of the cohomology theory E^* are given by

$$E^*(S^0) \cong \pi_{-*}F(S^0, E) \cong \pi_{-*}(E).$$

Example 7.12. Let us compute the coefficients of $KU^*(-)$, i.e. $KU^*(S^0) = \pi_{-*}KU$. We already computed $KU^0(S^0) = \mathbb{Z}$; therefore, $KU^{2n}(S^0) = \mathbb{Z}$ for all $n \in \mathbb{Z}$. We need only compute $KU^1(S^0) = \widetilde{K}(S^1)$:

$$\widetilde{K}(S^1) \cong [S^1, BU \times \mathbb{Z}]_* \cong \pi_1(BU) = 0.$$

In conclusion, the spectrum KU is even periodic, in the following sense:

$$\pi_n(KU) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Another consequence of Bott periodicity is that $KU \simeq \Omega^2 KU$, or $\Sigma^2 KU \simeq KU$ in other words.

Remark 7.13. Let us think about duality for a second. By adjunction, $E \wedge X$ and $F(X, E)$ correspond to each other. The covariant functor $E \wedge -$ is E-homology, and the contravariant functor $F(-, E)$ is E-cohomology. What about the covariant functor $F(E, -)$? If $E = \mathbb{S}$, then this would be homotopy; is it worth it to call $F(E, -)$ "E-homotopy"?

Let's get back to spaces for a second. Then we have this other duality: just as spheres have simple cohomology (and hard homotopy) and co-represent homotopy, Eilenberg–Mac Lane spaces have simple homotopy (and hard cohomology) and represent cohomology. From the former you build all CW-complexes, and from the latter you can make Postnikov towers. So they are all very fundamental, and from this point of view, cohomology is actually dual to homotopy. Homology is a bit funky from this perspective: if you want to get it from Eilenberg–Mac Lane spaces, you can't do better than $\widetilde{H}_n(X; G) \cong \operatorname{colim}_k \pi_{n+k}(K(G, k) \wedge X)$.

Finally, recall that if k is a field, then $H^*(X; k)$ is the vector space dual to $H_*(X; k)$.

This is a lot to digest, and it's not clear how it all glues together cleanly. Let me just offer you some interesting reads. MO: What is homology, anyway? (with answers by Lurie, Scholze and Shulman), the very abstract nLab: cohomology, the more accessible nLab:Eckmann–Hilton duality, and n-Category café: Cohomology and homotopy (with lots of interesting comments).

Notice how (co)homology theories, by definition, invert suspension, in a sense. Since spectra are meant to universally invert the suspension functor[2], we expect there to be a dotted arrow making the following diagram commute, for any spectrum E:

$$\begin{array}{ccc} CW_* & \xrightarrow{E_n(-)} & \mathrm{Ab} \\ {\scriptstyle \Sigma^\infty}\big\downarrow & \nearrow & \\ \mathrm{Sp} & & \end{array}$$

This is what we have seen above: the dotted arrow is none other than $\pi_n(E \wedge -)$. If we hadn't introduced the smash product in Section 4, we wouldn't be able to write such a nice diagram.

We can now wonder about the following: why precompose $\pi_n(E \wedge -)$ with Σ^∞?

Definition 7.14. Let E be a spectrum. For $n \in \mathbb{Z}$, define $E_n(-) : \mathrm{Sp} \to \mathrm{Ab}$ as $\pi_n(E \wedge -)$, and $E^n(-) : \mathrm{Sp} \to \mathrm{Ab}$ as $\pi_{-n} F(-, E)$.

If $E = H\mathbb{Z}$, we denote $E_n(X)$ by $H_n(X)$, and similarly for cohomology.

Thus, for $X \in \mathrm{Top}_*$, we have $E_n(X) = E_n(\Sigma^\infty X)$ by definition.

There is an axiomatic definition of (co)homology theories on spectra, similar to that on pointed CW-complexes. See [**Rud98**, 3.10]. You'd then prove that E_* and E^* satisfy these axioms.

And once we have such definitions, we can finally correctly formulate the question posed at the end of Section 1. Let $h^* : \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{GrAb}_{\mathbb{Z}}$ be a cohomology theory in spectra. Then there exists a spectrum E such that $h_* \cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp})}(-, E)_{-\bullet}$. This is enriched representability

[2] You are probably wondering: did we actually prove that? Not really, we didn't. But it's doable with some more technology, see Section 2.

of h^*, modulo the minus sign, which was just a matter of convention.[3] A full proof, which uses Brown representability, can be found in [**Rud98**, III.3.21].

Remark 7.15.
(1) If you stop and think about what we are doing, your head may start to spin: so, we defined spectra thinking of them as (objects representing) cohomology theories. Cohomology theories are functors applied to spaces. But now we're saying that we can think of cohomology theories applied to spectra. A cohomology theory applied... to a cohomology theory!? Moreover, switching to homology for a second, if E and X are spectra, $E \wedge X \simeq X \wedge E$ can be thought of as the E-homology object of X, or as the X-homology object of E... What can I say? Such is the nature of spectra.
(2) Expanding on these thoughts, you would get a proof that the stable homotopy category is equivalent to the category of cohomology theories on spectra. Is it equivalent to the category of cohomology theories on pointed spaces/CW-complexes? No, it is not. There exist maps of spectra $E \to F$ which are not nullhomotopic, but whose induced map on cohomology theories is zero. So while the correspondence spectra – (co)homology theories works pretty good, this is not the case for their maps, and therefore their categories. This is closely related to the existence of *phantom maps*, see e.g. [**Rud98**, Chapter 3].

We finish with some words from Eric Peterson: "Spectra are an enrichment of homology theories where homotopy theory can be done".

2. The abstract construction of the stable homotopy category by Boardman

Boardman's was one of the first really satisfactory constructions for the stable homotopy category. It used high-level category theory at a time when category theory was still dismissed as "abstract nonsense" by many mathematicians; perhaps that's the reason Boardman's approach did not gain much traction, especially because in the same manuscript he described a more concrete approach, later further developed by Adams.

The starting observation is the following. Every abelian group is the filtered colimit of its finitely presented abelian subgroups. Moreover, the finitely presented abelian groups are exactly the compact objects of Ab. Here, an object $c \in \mathcal{C}$ is *compact*, also called "finitely presentable", if $\mathrm{Hom}_{\mathcal{C}}(c, -)$ preserves filtered colimits.[4] These properties can be summarized by saying that Ab is a locally presentable category whose compact objects are the finitely presented abelian groups.

This implies another statement: Ab is equivalent to the ind-completion of its full subcategory of finitely presented abelian groups.[5] The *ind-completion* of a category is the formal, universal adjunction of filtered colimits to it, see [**KS06**, Chapter 6], in particular 6.3.4 proves the statement we just made. Similarly, the ind-completion of the category of finite (simplicial) sets is equivalent to the category of all (simplicial) sets.[6]

[3] For the graded hom, we chose a sign that privileged homotopy, as in Corollary 6.8, but we could just as well have put the opposite sign, and we would have gotten actual representability here, privileging cohomology.

[4] The terminology is a bit unfortunate in that compact objects in Top are not the compact topological spaces! See MO/288648: the compact objects in spaces are the finite discrete spaces. This is related to the subtleties on sequential colimits considered in Section 1.5. On the other hand, the compact objects in simplicial sets *are* the finite simplicial sets, whose geometric realizations are (classical) compact spaces.

[5] It's also equivalent to the ind-completion of finitely generated abelian groups.

[6] For spaces this doesn't work because of the previous footnote. This can also be phrased as "Top is not a locally presentable category", which is one justification for preferring the category of simplicial sets to that of topological spaces. For CW-complexes this also doesn't work. While it is true that every CW-complex is the filtered colimit of its finite CW-subcomplexes, if you take e.g. the comb space, this is the sequential colimit of finite CW-complexes but it is not a CW-complex (it is not locally path-connected), so the ind-completion of the category of finite CW-complexes

Bottom line: nice enough categories can be recovered from a subcategory of "finite" objects via ind-completion. So, if we can build a satisfactory category of *finite spectra*, then we can take its ind-completion as a definition for the category of spectra.

Finite spectra, which made a quick appearance in Section 3, can in turn be built categorically: start with pointed, finite CW-complexes CW_*^{fin}, and invert the suspension functor: that is, take the colimit of[7]

$$CW_*^{\text{fin}} \xrightarrow{\Sigma} CW_*^{\text{fin}} \xrightarrow{\Sigma} CW_*^{\text{fin}} \xrightarrow{\Sigma} \cdots$$

This gives a category Sp^{fin}, and its ind-completion is a model for Sp. See [**Vog70**] for the original description by Boardman, [**LMSM86**, Preamble] for a leisurely description of it, or [**Lur17**, 1.4.(B)] for a more modern approach via ∞-categories where Lurie also explains how this is equivalent to the limit of

$$\cdots \longrightarrow CW_* \xrightarrow{\Omega} CW_* \xrightarrow{\Omega} CW_*$$

where "limit" is taken in an appropriate higher-categorical sense; this presentation is closer to spectra being "infinite loop spaces".

There is a natural question to be asked, here: why not bypass the ind-completion completely and just invert the suspension functor on CW_*? Wouldn't that give us a model for Sp directly? Alas, that doesn't quite work. We can do that, and it gives us something called the Spanier–Whitehead category. It is important in its own right, but it's actually too small to be the category we really care about, e.g. it doesn't have all coproducts; Brown representability breaks down.[8] It does have historical importance (it predates the category of spectra), and is sometimes useful. A classical development of it can be found in [**Mar83**]; a more high-brow development of it can be found in [**Lur18**, C.1.1].

3. Some results around Eilenberg–Mac Lane spectra

Generalities. We already mentioned that the homotopy category of spectra is triangulated. More is true: it has a *t-structure*. Instead of defining that precisely, let me mention a couple of its features in this particular case. Just as we have connective spectra, we have *coconnective spectra*, whose positive homotopy groups vanish. These two classes of spectra interact in a pleasant way; an important feature here is that the *heart* of this structure, that is, the subcategory of spectra which are both connective and coconnective, is an abelian category. This is actually true in greater generality, e.g. for any stable model/∞-category with a t-structure [**Lur17**, 1.2.1.12].

Let's be more explicit. In this case, the abelian category is none other than the category of abelian groups, and the precise statement is that the functor

$$\pi_0 : \text{Hom}_{\text{Ho(Sp)}}(X, Y) \to \text{Hom}_{\text{Ab}}(\pi_0 X, \pi_0 Y) \tag{7.16}$$

is an isomorphism when X is connective and Y is coconnective. In particular,

$$\pi_0 : \text{Hom}_{\text{Ho(Sp)}}(HA, HB) \to \text{Hom}_{\text{Ab}}(A, B)$$

is an isomorphism; its inverse is H, and thus $H : \text{Ab} \to \text{Ho(Sp)}$ is fully faithful. Note that the restriction of $\pi_0 : \text{Ho(Sp)} \to \text{Ab}$ to the subcategory of spectra with homotopy concentrated in

is bigger than the category of CW-complexes. Since every space is weakly equivalent to a CW-complex, we'd like to say that the ind-completion of CW^{fin} is equivalent in a homotopical sense to the category of spaces. This is true e.g. ∞-categorically, as mentioned in this lecture of Lurie.

[7]If you don't see how this colimit should invert Σ, do this: let A be an abelian group, let $n \in \mathbb{Z}$, and take the colimit of the multiplication-by-n map: prove it is isomorphic to the localization $A[1/n]$. Also, note that this is a colimit only in an appropriate, higher-categorical sense.

[8]Reader beware: some authors call "Spanier–Whitehead category" what we have called Sp^{fin} above...

degree 0 is an equivalence of categories. Finally, note that this proves uniqueness of Eilenberg–Mac Lane spectra, in the sense that if X is a spectrum whose only non-trivial homotopy group is $\pi_0(X) = A$, then X is weakly equivalent to HA. Indeed, by (7.16) we get a unique morphism $f : X \to HA$ in $\mathrm{Ho}(\mathrm{Sp})$ which induces an isomorphism in π_0, hence it's an isomorphism in $\mathrm{Ho}(\mathrm{Sp})$ since both X and HA are connective and coconnective.

See [**Sch**, 5.24] for a proof of all this in the context of symmetric spectra; this is also in [**Lur17**, 1.2] in the context of ∞-categories.

Side remark 7.17. In Side remark 3.43, we mentioned a certain connective spectrum denoted ku, called the spectrum of *connective topological complex K-theory*. One can alternatively define it as the connective cover of KU, i.e. ku is characterized by its being connective and there being a map $ku \to KU$ which is an equivalence on non-negative homotopy groups. One concrete construction of connective covers is in [**Rud98**, II.4.15].

The Hurewicz theorem. The Eilenberg–Mac Lane spectrum $H\mathbb{Z}$ plays a special role, as we already know from the special role played by ordinary integral homology, witnessed by the Hurewicz theorem most notably. One possible abstract reason for its importance is that it's the first non-trivial truncation of the sphere spectrum: similarly as in the previous section, the inclusion of coconnective spectra into spectra has a left adjoint, and that functor takes the sphere spectrum $\mathbb{S}$ to $H\mathbb{Z}$. The unit of that adjunction is a map of spectra $\mathbb{S} \to H\mathbb{Z}$.

We can build that map very explicitly, though. For any $n \geq 0$, we can choose maps $S^n \to K(\mathbb{Z}, n)$ which correspond to $1 \in \mathbb{Z} \cong \pi_n(K(\mathbb{Z}, n))$ and are compatible with the structure maps. This defines a map of spectra $\tau : \mathbb{S} \to H\mathbb{Z}$ called the *Hurewicz morphism*.

If $n \in \mathbb{Z}$, a spectrum X is called *n-connected* if $\pi_k(X) = 0$ for all $k \leq n$. We can now state the stable Hurewicz theorem:

Theorem 7.18 (Stable Hurewicz)**.** *Let $n \in \mathbb{Z}$ and X be an n-connected spectrum. Then the integral homology groups of X below level $n+1$ are trivial, and $\tau \wedge \mathrm{id}$ induces an isomorphism of abelian groups*

$$\pi_{n+1}(X) \cong \pi_{n+1}(\mathbb{S} \wedge X) \xrightarrow[(\tau\wedge\mathrm{id})_*]{\cong} \pi_{n+1}(H\mathbb{Z} \wedge X) \cong H\mathbb{Z}_{n+1}(X).$$

There are different ways to prove this. The one adopted by [**Sch**, II.6.30] is a direct consequence of the second paragraph of Remark 6.16: since X is n-connected and $H\mathbb{Z}$ is (-1)-connected, this gives an isomorphism $\pi_0(H\mathbb{Z}) \otimes \pi_{n+1}(X) \to \pi_{n+1}(H\mathbb{Z} \wedge X)$, and one checks that this is precisely the isomorphism above.

Another way to prove it passes via the theory of *finite spectra*, which unfortunately we didn't have the time to explore in these notes. Just as any abelian group is the filtered colimit of its finite subgroups, and any CW complex is the filtered colimit of its finite subcomplexes, we have that any cell spectrum is the filtered colimit of its finite cell subspectra.[9] But we didn't define what's a finite spectrum. There are different characterizations; the most concrete one is probably that it's a spectrum isomorphic in $\mathrm{Ho}(\mathrm{Sp})$ to one of the form $\Sigma^n\Sigma^\infty K$ where K is a finite pointed CW-complex and $n \in \mathbb{Z}$. More abstractly, the finite spectra are precisely the small objects in $\mathrm{Ho}(\mathrm{Sp})$, where X is *small* if $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp})}(X, -)$ preserves direct sums. For details and other characterizations, see [**Sch**, II.7.2]. The fact that any cell spectrum is the filtered colimit of its finite cell subspectra is proven similarly as for spaces; a reference is [**EKMM97**, III.2.3]. Combining this approximation of a spectrum by finite spectra with the Hurewicz theorem for spaces, you can prove stable Hurewicz. This was mentioned in this m.SE answer, where a pointer to another more hands-on proof is also given. The latter hands-on proof is explained in full detail in [**Rud98**, II.4.7] in the setting of Adams' CW-spectra.

[9] You may want to read up on locally presentable categories, and/or on combinatorial model categories.

Bockstein morphisms. As promised at the end of Section 2, we now give examples of maps which in classical algebra are injective or surjective, but when transported to the world of spectra, they have interesting fibers and cofibers.

If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then $HA \to HB \to HC$ is a cofiber sequence of spectra. This is proven in [**Sch**, 5.28]. The cofiber map $\delta : HC \to \Sigma HA$ is called the *Bockstein morphism*.

If X is a pointed space, this gives a natural morphism in homology (and similarly in cohomology)

$$\widetilde{H}_n(X; C) \to \widetilde{H}_{n-1}(X; A).$$

This is the classical Bockstein homomorphism that you may already be acquainted with, e.g. from [**Hat02**, 3.E]. A particularly important example is given by the sequence

$$0 \to \mathbb{Z}/p \xrightarrow{\cdot p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0,$$

in which case the classical Bockstein homomorphism is given by maps

$$\widetilde{H}_n(X; \mathbb{Z}/p) \to \widetilde{H}_{n-1}(X; \mathbb{Z}/p).$$

In conclusion, the map $HC \to \Sigma HA$ is really the spectral version of the Bockstein homomorphism.

Other results. See [**Sch**, II.6.2] and [**EKMM97**, Chapter IV] for many other elementary results of spectra which generalize known results in spaces, like the Künneth theorem or the universal coefficient theorem (of which we will say a bit in Proposition 7.21). We quickly mention [**Sch**, 6.30.ii] which generalizes the homology Whitehead theorem. If $f : X \to Y$ is a morphism of bounded-below spectra (this means that below a certain level, the homotopy groups are zero), then f is a weak equivalence if and only if it is an integral homology equivalence, i.e. it induces isomorphisms on $H\mathbb{Z}_*$.

4. Moore spectra

Recall that a *Moore space* of type (G, n), denoted $M(G, n)$, is a pointed CW-complex that has the property that all its reduced integral homology groups are zero except in degree $n \geq 1$ where it is an abelian group G. They are, in a sense, the integral homology counterparts to Eilenberg–Mac Lane spectra. See [**Hat02**, 2.40].

The easiest Moore space to construct is $M(\mathbb{Z}/m, n)$. Indeed, take a degree m map $f_m : S^n \to S^n$, and define $M(\mathbb{Z}/m, n)$ to be its homotopy cofiber. In other words, we are attaching a D^{n+1} to S^n via a degree m map. The result follows by the long exact sequence in homology associated to this cofiber sequence. You could say that we have realized the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}/m \to 0$ homologically in degree n as the cofiber sequence $S^n \xrightarrow{f_m} S^n \to M(\mathbb{Z}/m, n)$. Having this example, and realizing that S^n is an $M(\mathbb{Z}, n)$, we can build all the $M(G, n)$ where G is finitely generated, and for the general case you need to take a presentation of G.

Now, Moore spaces allow us to introduce coefficients in homology at the space level:

$$\widetilde{H}_n(X; G) \cong \widetilde{H}_n(X \wedge M(G, n); \mathbb{Z}).$$

This follows from the Künneth theorem [**Hat02**, Page 277].

A construction of $H_*(-; G)$ is no harder than that of $H_*(-; \mathbb{Z})$ (say, in singular homology), but what if we want to introduce coefficients in an arbitrary homology theory? Then the road via Moore spectra seems to be more reasonable.

Proposition 7.19. *Let G be an abelian group. There exists a spectrum SG, the* Moore spectrum *of G, satisfying the following properties:*

(1) SG is connective,
(2) $\pi_0(SG) = G = H_0(SG;\mathbb{Z})$,
(3) $H_i(SG) = 0$ for all $i \neq 0$.

These properties determine SG uniquely up to equivalence.[10]

Example 7.20. $\mathbb{S}$ is an $S\mathbb{Z}$. The spectrum $S\mathbb{Z}/p$ can be built like this: take a map $\mathbb{S} \to \mathbb{S}$ of degree p in Ho(Sp), something we have already constructed in different ways, and complete it to a distinguished triangle. Now take the third term in the sequence, which is usually denoted by $\mathbb{S}/p$.

In general, you can build an SG simply as $\Sigma^{\infty-n}M(G,n)$, as you can very easily check.

Now, if E is a spectrum and G is an abelian group, we can call $E \wedge SG$ the *spectrum of E-homology with coefficients in G*. To justify this, we can state the following proposition, which is not hard to prove from the description of SG as fitting in a distinguished triangle.

Proposition 7.21 (Universal coefficient theorem)**.** *Let X and E be spectra and G be an abelian group. For every $n \in \mathbb{Z}$ there is a short exact sequence*

$$0 \longrightarrow E_n(X) \otimes G \longrightarrow (E \wedge SG)_n(X) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(E_{n-1}(X), G) \longrightarrow 0$$

which need not split. In particular, if G is torsion-free, then $(E \wedge SG)_n(X) \cong E_n(X) \otimes G$.

There is a similar theorem for cohomology, but you need to add some finiteness assumptions for it to work, see [Ada74, III.6.6].

Example 7.22. If $E = \mathbb{S}$ and $G = \mathbb{Z}/p$, then we have short exact sequences

$$0 \longrightarrow \frac{\pi_n(X)}{p\pi_n(X)} \longrightarrow (\mathbb{S}/p)_n(X) \longrightarrow {}_p\pi_{n-1}(X) \longrightarrow 0$$

where ${}_pA$ denotes the subgroup of p-torsion elements (those $a \in A$ such that $pa = 0$).

You can see how Moore spectra are related to the question of topological realization of algebraic operations on homotopy groups (or on more general homology groups). We can ask ourselves: can we realize the quotients $\pi_n(X)/p\pi_n(X)$ as $E_n(X)$ for some E? What about the $\pi_n(X)[1/p]$? Or the localizations $\pi_n(X)_{(p)}$? We could also have asked those question for spaces, by the way. These are important questions but we shall not address them here, see [Bou79], or [MP12, Part 2] for the theory for spaces, or [Sch, II.9] for the theory for spectra.

An important case is worth mentioning: when R is a localization of $\mathbb{Z}$ at a set of primes, then the localization of a spectrum X at R is $X \wedge SR$. For example, if we take $R = \mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at p (invert all other primes), then the localization of X at p is $X \wedge S\mathbb{Z}_{(p)}$. It satisfies that

$$\pi_*(X \wedge S\mathbb{Z}_{(p)}) \cong \pi_*(X) \otimes \mathbb{Z}_{(p)} \cong \pi_*(X)_{(p)}.$$

Going forward, other, more refined localizations are part of a bigger problem to describe the stable stems: that of chromatic homotopy theory. See [BB20] for an introduction.

Example 7.23. One advantage of seeing topological K-theory as a spectrum is that we can use all the operations that are available to us. For example, we could use Moore spectra to introduce coefficients. Or we could easily build a mod p version: take the degree p map $p : \mathbb{S} \to \mathbb{S}$ and smash it with KU, getting $p : KU \simeq KU \wedge \mathbb{S} \xrightarrow{\mathrm{id} \wedge p} KU \wedge \mathbb{S} \simeq KU$. Now take its cofiber, which you can denote KU/p. Something interesting happens to this when p is an odd prime: it splits,

$$KU/p \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} K(1)$$

[10] But S is not really a functor, in general. See [Sch, II.6.44] for details.

where $K(1)$ is an important spectrum called the *1-st Morava K-theory spectrum at the prime p*. This was first proven in [**Ada69**] and is a foundational result of *chromatic homotopy theory*, see [**BB20**] for an introduction.

5. Thom and cobordisms

5.1. Spherical fibrations. We can turn real vector bundles into a more topological object: a spherical fibration. It suffices to one-point compactify all the fibers: if $\mathbb{R}^n \to E \to B$ is a vector bundle, then we can build a Hurewicz fibration $\mathrm{Sph}(E) \to B$ whose fibers are homotopy equivalent to S^n; we shall write this as $S^n \to \mathrm{Sph}(E) \to B$.

Note, moreover, that the fibration thus obtained has a section: choosing the points at infinity gives us a map ∞

$$S^n \longrightarrow \mathrm{Sph}(E) \underset{\infty}{\overset{p}{\rightleftarrows}} B$$

such that $p \circ \infty = \mathrm{id}_B$. So let us forget about vector bundles for now and think of objects as above: spherical fibrations with a section, or, following the terminology of [**Rud98**], *sectioned spherical fibrations*. The examples of vector bundles and sectioned spherical fibrations are two examples of a more general theory, see [**Rud98**, Chapter IV].

We have already recalled that the set of isomorphism classes of n-plane complex vector bundles over B is in bijection with $[B, BU(n)]$. An analogous theorem is true for sectioned spherical fibrations. First, define a *sectioned homotopy over B* from $E_1 \to B$ to $E_2 \to B$ to be a homotopy of maps $E_1 \to E_2$ such that at any time t it is a map over B, and it is compatible with the sections. The resulting notion of homotopy equivalence is called *sectioned fiber homotopy equivalence*; note that indeed the induced maps on fibers are homotopy equivalences as well. Dold and Lashof [**DL59**] remarked that this is equivalent to it. In fact, if both maps over B are fibrations, then a map $E_1 \to E_2$ is a fiber homotopy equivalence as soon as it is a homotopy equivalence [**May99a**, Page 52].

Let $F(n)$ denote the topological monoid of pointed homotopy self-equivalences of S^n, then the set of sectioned fiber homotopy equivalence classes of sectioned spherical fibrations over B is in bijection with $[B, BF(n)]$, via the existence of a universal sectioned spherical fibration. [11] The classification of fibrations is harder than the classification of locally-trivial bundles and took some years to set up in full. An important early reference is [**May75**].

We have explained above how to construct the Whitney sum of vector bundles. There is an analogous construction for sectioned spherical fibrations, where the fibers get smashed: given two of them $S^n \to E \to B$ and $S^m \to E' \to B$, you can build a a new sectioned spherical fibration $S^{n+m} \cong S^n \wedge S^m \to E \oplus E' \to B$ [**Rud98**, IV.1.4(g)]. We also have trivial sectioned spherical fibrations $S^n \to X \times S^n \xrightarrow{\varepsilon^n} X$.

Similarly to $\mathrm{Vect}(X)$, we have a commutative monoid $\mathcal{S}(X)$ of sectioned spherical fibrations and we can take its Grothendieck group. It is denoted $K_F(X)$ e.g. in [**Ada78**]. Analogously for K-theory, we can define $\widetilde{K}_F(X)$ for pointed X and, if X is compact, then any sectioned spherical fibration has a complement, and $\widetilde{K}_F(X)$ is isomorphic to the group of stable fibrewise homotopy equivalence classes of sectioned spherical fibrations over X. When X is moreover connected, this is isomorphic to $[X, BF]$. Here BF is built as follows: consider the maps $F(n) \to F(n+1)$, which take a pointed homotopy equivalence $S^n \to S^n$ to its suspension. This gives maps $BF(n) \to BF(n+1)$. Define $BF = \mathrm{colim}_n BF(n)$.

[11] I think the letter F comes from "fibration", to remind us that we are classifying fibrations instead of the more structured locally-trivial maps.

When X is not connected, one can do something similar to Remark 3.57, so we are going to call any map $X \to BF$ a *stable spherical fibration*.[12] Alternatively, we can think about it as a virtual spherical fibration of dimension zero over every point. With more technology, we would be able to justify why we can also think of it as a map to X whose fibers are sphere spectra up to weak equivalence, see e.g. the introduction to [ABG+14a]. In general, for compact X we have $K_F(X) \cong [X, BF \times \mathbb{Z}]$.

The real K-theory group $K_O(X)$ and $K_F(X)$ are related by the construction given at the beginning of the section. We can build a map of commutative monoids $\mathrm{Vect}_{\mathbb{R}}(X) \to \mathcal{S}(X)$ which takes a real vector bundle and compactifies all the fibers, so we get an induced map on Grothendieck groups $K_O(X) \to K_F(X)$ which induces $J : \widetilde{K}_O(X) \to \widetilde{K}_F(X)$. This is one instance of the famous *J-homomorphism* of Adams.

Proposition 7.24. *The homotopy groups of BF are:*

$$\pi_i(BF) = \begin{cases} 0 & \text{if } i = 0 \\ \mathbb{Z}/2 & \text{if } i = 1 \\ \pi_{i-1}\mathbb{S} & \text{if } i \geq 2. \end{cases}$$

PROOF. From the definition of B, we have $\pi_0(BF) = 0$.If $i \geq 1$, then

$$\pi_i(BF) = \pi_i \mathrm{colim}_n BF(n) \cong \mathrm{colim}_n \pi_i BF(n) \cong \mathrm{colim}_n \pi_{i-1} F(n).$$

Note that $F(n) \simeq \Omega^n_{\pm 1} S^n$, i.e. the union of the two connected components of $\Omega^n S^n$ corresponding to the maps of degree 1 or -1. Therefore,

$$\pi_1(BF) \cong \mathrm{colim}_n \pi_0(F(n)) \cong \mathbb{Z}/2.$$

If $i \geq 2$, then

$$\pi_i(BF) \cong \mathrm{colim}_n \pi_{i-1} F(n) \cong \mathrm{colim}_n \pi_{i-1} \Omega^n_{\pm} S^n \overset{i\geq 2}{\cong} \mathrm{colim}_n \pi_{i-1} \Omega^n S^n \cong \mathrm{colim}_n \pi_{i-1+n} S^n = \pi_{i-1}\mathbb{S}.$$

□

An important particular case of the J-homomorphism appears when $X = S^n$, for $n \geq 2$. Then $J : \widetilde{K}_O(S^n) \to \widetilde{K}_F(S^n)$. The domain is

$$\widetilde{K}_O(S^n) \cong [S^n, BO] \cong \pi_n(BO) \cong \pi_{n-1}(O),$$

whereas the codomain is

$$\widetilde{K}_F(S^n) \cong [S^n, BF] \cong \pi_n(BF) \cong \pi_{n-1}(\mathbb{S}).$$

So we get maps

$$J : \pi_n(O) \to \pi_n(\mathbb{S}), \qquad n \geq 1.$$

The domain is completely computed (that's the real Bott periodicity theorem). Computing the codomain is hard, as we already know, but it turns out that the image of this map, the so-called *image-of-J*, is computable. This is foundational work of Adams and Quillen: it computes a chunk of the stable stems. Note how the strategy changes: we don't proceed homotopy group by homotopy group. A good summary can be found in [Mat12]. This is one of the first steps in the computational program of chromatic homotopy theory, which we have already mentioned in passing.

Another important instance of the J-homomorphism is given at the level of the representing spaces. For example, you have a map $O(n) \to F(n)$ which takes an isometry of $\mathbb{R}^n$ and one-point compactifies it, giving you a basepoint-preserving homotopy equivalence of S^n. This is

[12] You might be wondering what happened to the sections in this terminology. It turns out they become irrelevant in the stable setting... See [Rud98, IV.4.24].

compatible with the structure maps on both sides, so taking colimits and applying B we get a map $BO \to BF$ also called the J-homomorphism.

Side remark 7.25. The $\mathbb{Z}/2$ in $\pi_1(BF)$ should be understood as the units of $\mathbb{Z}$, i.e. $GL_1(\mathbb{Z})$. The space BF (which is an infinite loop space, hence, equivalently, a connective spectrum, as mentioned e.g. in Side remark 4.15) is a spectral enhancement of that, under realization of $\mathbb{Z}$ as $\pi_0(\mathbb{S})$. A bit more precisely, consider the canonical map $QS^0 \to \pi_0(QS^0) \cong \pi_0\mathbb{S} \cong \mathbb{Z}$ which takes a point to its connected component, and consider the inclusion of the units $\mathbb{Z}/2 \cong GL_1(\mathbb{Z}) \to \mathbb{Z}$. Now take the homotopy pullback of these two maps to $\mathbb{Z}$ in Top: you get a space denoted $GL_1(\mathbb{S})$ which is equivalent to F. This is a result from the late seventies by May and collaborators, but see [**ABG**+**14a**] and the references therein. $GL_1(\mathbb{S})$ is, in other words, the union of the components of QS^0 which correspond to units in $\pi_0(\mathbb{S})$.

On a different note, recall Side remark 3.43: the connective complex K-theory spectrum ku is the connective spectrum associated to the infinite loop space $BU \times \mathbb{Z}$, and it is obtained starting from the symmetric monoidal topological category of finite-dimensional complex vector spaces. We could do something similar here. Consider the category whose objects are pointed spheres, and whose maps are based homotopy equivalences. It is topological and symmetric monoidal with $\wedge$. The machine begets the E_∞-space $\bigsqcup_n BF(n)$, whose group completion is the infinite loop space $BF \times \mathbb{Z}$, whose associated connective spectrum is also known as the Picard spectrum of $\mathbb{S}$, whose connected components are equivalent to BF.

5.2. Thom spaces.

Definition 7.26. The *Thom space* of a sectioned spherical fibration

$$S^n \longrightarrow X \underset{\infty}{\overset{p}{\rightleftarrows}} B$$

is defined to be $X/\infty(B)$, i.e. the (strict) cofiber $B \xrightarrow{\infty} X \to \mathrm{Th}(p)$. It is pointed by $[\infty(b)]$ for any $b \in B$.

So the Thom space identifies all the points at infinity.

For simplicity, the Thom space of the sectioned spherical fibration associated to a real vector bundle ξ is just denoted $\mathrm{Th}(\xi)$. When B is compact, then $\mathrm{Th}(\xi)$ can be identified with the one-point compactification of X.

Example 7.27. The Thom space of a sectioned spherical fibration is a generalization of the iterated suspensions of a space. Indeed, the Thom space of the sphere bundle associated to the trivial n-plane vector bundle over B is homeomorphic as a pointed space to $\Sigma^n B_+$.

The intuition is that a sectioned spherical fibration is a twist of B, and that the Thom space is like a suspension of B, twisted accordingly.

So the suspension isomorphism in this trivial case gives us that

$$\widetilde{H}^*(\mathrm{Th}(\mathrm{Sph}(B \times \mathbb{R}^n \to B))) \cong H^{*-n}(B).$$

Now, what happens to the above isomorphism when we consider any sectioned spherical fibration? To give a full answer to that would be too much of a detour, so we will be very quick about it. There is a theory of orientations[13] of sectioned spherical fibrations. For example, the (spherical fibration associated to the) tangent bundle of a manifold is orientable if and only if the manifold is orientable.

Theorem 7.28 (Thom isomorphism)**.** *If $S^n \to X \xrightarrow{p} B$ is an orientable sectioned spherical fibration, then there is an isomorphism*

$$\widetilde{H}^{*+n}(\mathrm{Th}(p)) \cong H^*(B).$$

[13] $H\mathbb{Z}$-orientations: they can be generalized to any ring spectrum.

5.3. Thom spectra. Let $f : X \to BF$ be a map, i.e. a stable spherical fibration, as we have explained before. Let X_n be $f^{-1}(BF(n))$. Define $f_n : X_n \to BF(n)$ as $f_n(x) = f(x)$. Thus, we have a commutative ladder like this:

$$\begin{array}{ccccccccccccc} X_0 & \xhookrightarrow{i_0} & X_1 & \xhookrightarrow{i_1} & \cdots & \hookrightarrow & X_n & \xhookrightarrow{i_n} & X_{n+1} & \hookrightarrow & \cdots & \hookrightarrow & X \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & & & \downarrow{\scriptstyle f} \\ BF(0) & \longrightarrow & BF(1) & \longrightarrow & \cdots & \longrightarrow & BF(n) & \longrightarrow & BF(n+1) & \longrightarrow & \cdots & \longrightarrow & BF \end{array}$$

Now, consider the universal sectioned n-spherical fibration $\gamma_{n+1} : E_{n+1} \to BF(n+1)$. Pull it back along f_{n+1}, and denote the corresponding fibration by $f_{n+1}^*(\gamma_{n+1}) : P_{n+1} \to X_{n+1}$. Pull it back along i_n:

$$\begin{array}{ccccc} P_n \oplus (X_n \times S^1) & \xrightarrow{h_n} & P_{n+1} & \longrightarrow & E_{n+1} \\ {\scriptstyle f_n^*(\gamma_n)\oplus\epsilon^1}\downarrow & \lrcorner & \downarrow{\scriptstyle f_{n+1}^*(\gamma_{n+1})} & \lrcorner & \downarrow{\scriptstyle \gamma_{n+1}} \\ X_n & \xrightarrow[i_n]{} & X_{n+1} & \xrightarrow[f_{n+1}]{} & BF(n+1) \end{array}$$

Exercise 7.29. *Prove the claim in the above diagram: namely, if you take the pullback on the left, then you can identify it with that Whitney sum. Hint: first prove it for the universal case, namely*

$$\begin{array}{ccc} E_n \oplus (BF(n) \times S^1) & \longrightarrow & E_{n+1} \\ {\scriptstyle \gamma_n\oplus\epsilon^1}\downarrow & \lrcorner & \downarrow{\scriptstyle \gamma_{n+1}} \\ BF(n) & \longrightarrow & BF(n+1). \end{array}$$

Here's another observation: if $p : E \to B$ is a sectioned spherical fibration, then

$$\mathrm{Th}(p \oplus \epsilon^1) \cong \Sigma \mathrm{Th}(p).$$

In this context, it's an exercise in [**Rud98**, IV.5.5]; in the context of vector bundles, it's explained in [**Swi75**, Page 229].

Now that we have these two observations at hand, we can make the following definition:

Definition 7.30. The *Thom spectrum* Mf of a stable spherical fibration $f : X \to BF$ is defined as follows: $Mf_n := \mathrm{Th}(f_n^*(\gamma_n))$, and the structure maps are given by[14]

$$\Sigma Mf_n = \Sigma \mathrm{Th}(f_n^*(\gamma_n)) \cong \mathrm{Th}(f_n^*(\gamma_n) \oplus \epsilon^1) \xrightarrow{\mathrm{Th}(h_n)} \mathrm{Th}(f_{n+1}^*(\gamma_{n+1})) = Mf_{n+1}.$$

Example 7.31. It follows from the definitions that $M(* \to BF) \cong \mathbb{S}$.

Remark 7.32. The original definition of a Thom spectrum concerned maps $X \to BO$. The generalization above was suggested by Mahowald and first worked out by Lewis in [**LMSM86**]; there are also more modern, ∞-categorical approaches, most notably in [**ABG$^+$14a**]. This classical Thom spectrum of a map $X \to BO$ is equivalent to our Thom spectrum of its composition with the J-homomorphism $BO \to BF$.

Example 7.33. Some of the most important Thom spectra are the "universal" ones. We have the following chain of maps of spaces:

$$BSU \longrightarrow BU \longrightarrow BSO \longrightarrow BO \xrightarrow{J} BF$$

[14]There is a notation clash below: $\mathrm{Th}(h_n)$ does not mean the Thom space of some fibration called h_n, but rather the functor Th applied to the map of sectioned spherical fibrations h_n.

(1) We can take $MO := M(BO \xrightarrow{J} BF)$. Setting up everything carefully [Rud98, IV.5.12.(e)], we have that the n-th space is the Thom space of the universal real n-plane vector bundle, $MO_n \simeq \mathrm{Th}(E_n \to BO(n))$. Thom proved in 1954 that MO is actually a generalized Eilenberg–Mac Lane spectrum: $MO \simeq H\pi_*(MO)$, and

$$\pi_*(MO) \cong \mathbb{Z}/2[x_n : n \geq 2, n \neq 2^t - 1, |x_n| = n].$$

See [Koc96, 3.7.6], [Rud98, IV.6.2], [Pet19, 1.5.7, 1.5.8].

(2) Similarly, we can define $MU := M(BU \to BO \xrightarrow{J} BF)$. This is *not* a generalized Eilenberg–Mac Lane spectrum. It is a very important player in the aforementioned chromatic homotopy theory, and in the connection between algebraic topology and algebraic (formal) geometry of e.g. [Pet19].[15] It is fundamental to the notion of *complex-oriented cohomology theory* [Hop99], [Rud98, Chapter VII], closely related to the theory of Chern classes. For example, from this theory one gets maps of spectra $MU \to H\mathbb{Z}$ and $MU \to KU$, the latter of which appears in the formulation of the *Conner–Floyd isomorphism* relating the KU-cohomology and the MU-cohomology of a finite CW-complex.

It was proven by Milnor in 1960 and later improved by Quillen in 1967 that $\pi_*(MU) \cong \mathbb{Z}[y_{n+1} : n \geq 0, |y_{n+1}| = 2(n+1)]$ [Koc96, 4.4.13], [Pet19, 2.6.5].

Similarly to what happened with KU (see Example 7.23), the localization of MU at the prime p, which is $MU \wedge S\mathbb{Z}_{(p)}$ as we mentioned in Section 4, splits as a wedge of suspensions of a spectrum called BP, the *Brown-Peterson spectrum* at the prime p. This is another foundational result of chromatic homotopy theory. See [Rav86, 4.1.12], [Rud98, Page 415].

(3) We can also consider the oriented or "special" variants (hence the capital "S"). For example, the special orthogonal group $SO(n)$ of isometries of $\mathbb{R}^n$ that preserve orientation (i.e. that have determinant 1) is included in $O(n)$, thus getting the map $BSO \to BO$. You can produce an injection of $U(n)$ into $SO(2n)$. The group $BSO(n)$ classifies oriented n-plane real vector bundles. The special unitary group $SU(n)$ is definied similarly. This gives the Thom spectra MSU and MSO. These are also not weakly equivalent to generalized Eilenberg–Mac Lane spectra, even though [Ada74, Page 208] says MSO is, without proof. He was probably thinking of the localization of MSO at the prime 2, $MSO_{(2)}$, which *does* satisfy this [Rud98, IV.6.5].

(4) We end by quoting a surprising result by Mahowald and Hopkins. The Eilenberg–Mac Lane spectra $H\mathbb{Z}$ and $H\mathbb{Z}/p$ are equivalent to Thom spectra! There exists a map $\Omega^2 S^3 \to BO$ whose Thom spectrum is equivalent to $H\mathbb{F}_2$ (Mahowald); something similar is true at other primes (Hopkins), but for a map not even to BF, but to $BF_{(p)}$ (we didn't define this at all); finally, gathering them together, one gets that $H\mathbb{Z}$ is the Thom spectrum of a map to BF (Hopkins).

A classical, textbook proof of Mahowald's result can be found in [Rud98, IX.5.8]; a modern one encompassing the extensions by Hopkins is in [ACB19].

Exercise 7.34. *Let $f : X \to BF(n)$ be the classifying map for a sectioned spherical fibration. Prove that*

$$M(X \xrightarrow{f} BF(n) \to BF) \simeq \Sigma^{\infty - n}\mathrm{Th}(f).$$

[15] It is hard to understate this importance! See e.g. the paragraph "Why formal groups?" of section B.2 of Peterson's book, where the author concludes with "surprise and confusion" at the fact that this spectrum has its origins in manifold geometry. I share this feeling.

5.4. Unoriented cobordism. Some references: [**Sto68**], [**DK01**], [**Koc96**], [**Mil01**], [**Rud98**].

If M is a smooth manifold and $M \to BO(n)$ is a classifying map for its tangent bundle, then a lift of this map to $BSO(n)$ is equivalent to a choice of orientation on M; similarly, a lift of $M \to BO(2n)$ to $U(n)$ is akin to an almost-complex structure on M, etc. So you see how the theory of manifolds can enter the game, via their tangent bundles. Let's explore this further. The exposition will be sketchy as we intend to provide a mere glimpse: one could make a whole course or two on cobordism theory! "Manifold" will mean "compact smooth manifold with boundary", and $\varnothing$ is a manifold of all dimensions.

Definition 7.35. Let M, N be n-manifolds. A *cobordism*[16] between M and N is an $(n+1)$-manifold B such that $\partial B \cong M \sqcup N$. If there exists such a cobordism, we say M and N are *cobordant*.

More generally, if X is a space and $f : M \to X$, $g : N \to X$ are two maps, a *cobordism* between them is an $(n+1)$-manifold B and a map $F : B \to X$ such that $\partial B \cong M \sqcup N$ and F restricted to the boundary gives $f \sqcup g$. In this case, we say that f and g are *cobordant*.

To recover the former from the latter, just take $X = *$.

Proposition 7.36. *"Cobordism" is an equivalence relation, and the quotient $\Omega_n^O(X)$ of the set $\{M \to X : M$ n-manifold$\}$ by the cobordism relation is an abelian group of order 2, with sum given by the disjoint union, and neutral element given by $\varnothing \to X$.*

In particular, $\Omega_n^O(*)$ is the abelian group of equivalence classes of n-manifolds up to cobordism.

PROOF. Reflexivity: consider $M \times I$. Symmetry is easy. Transitivity: you need to use the collar neighborhood theorem to glue the cobordisms.

Associativity, commutativity and unitality of $\sqcup$ are easy. As for the inverses, if $f : M \to X$, consider $F : M \times I \to M \xrightarrow{f} X$. Then $\partial F \cong f \sqcup f$, so $[f]$ has an inverse and it is actually $[f]$, hence $\Omega^O(X)$ has order 2. ☐

Note that an element in $\Omega_n^O(*)$ is zero if it is *null-cobordant*, i.e. it is the boundary of some other manifold. An example of one such manifold is given by the real projective plane; see MO: Examples of manifolds that are not boundaries.

We can identify this group with something we have already introduced.

Theorem 7.37 (Pontryagin-Thom)**.** *For a space X, there is an isomorphism*

$$\Omega_*^O(X) \cong MO_*(X_+),$$

and thus cobordism is an unreduced homology theory. Taking $X = *$, we deduce that $\Omega_*^O(*) \cong \pi_*(MO)$.

Thus, the geometrical object given by the abelian group of n-manifolds up to cobordism gets identified with a stable-homotopy-theoretical object, $\pi_*(MO)$, which can be computed using homotopical tools.

Remark 7.38. Recall how singular homology is defined: you want to probe the space X with simplices Δ^n. The set of continuous functions $\Delta^n \to X$ is not an abelian group, so you take the free abelian group on this, and these fit into a chain complex of abelian groups out of which you take homology.

Now, suppose we want to probe X with manifolds, instead. Consider

$$A_n(X) = \{Z \to X : Z \text{ compact manifold with boundary of dimension } n\}.$$

[16] Also called *bordism*; confusingly, the prefix *co* does not signal a duality here.

This is a cancellative commutative monoid with the disjoint union as sum and the map $\varnothing \to X$ as unit. Note that the disjoint union of two such manifolds is again such a manifold, which didn't work for simplices: the disjoint union of simplices is not a simplex.

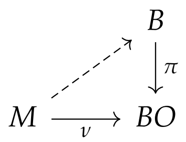
Now, define a boundary map $\partial : A_n(X) \to A_{n-1}(X)$. It takes $Z \to X$ to the restriction to the boundary, $\partial Z \to X$. It is a morphism of commutative monoids. Moreover, $\partial^2 = 0$, so this forms a chain complex of commutative monoids.

You can then observe that $\Omega_n^O(X)$ is the n-th homology of the above complex (properly interpreted).

This description of cobordism via chain complexes does not violate Remark 7.10, because these are not chain complexes of abelian groups but of commutative monoids. It is due to [**Koc78**], I learned about it from [**Pet19**].

5.5. Other cobordisms. We can generalize the cobordisms above, which are after all not too interesting from a homotopical perspective, since MO is a generalized Eilenberg–Mac Lane spectrum. For example, we may want to consider what happens when we take orientations into consideration. More generally:

Definition 7.39. Let $\pi : B \to BO$ be a map.[17] A *π-structure* on a manifold M is a homotopy class of lifts of a map $\nu : M \to BO$ classifying its stable normal bundle (recall Remark 3.54):



Remark 7.40. We could have considered the stable tangent bundle, see MO:141267. One reason to prefer the stable normal bundle is that it's generalizable to contexts other than manifolds which have stable normal bundles but no stable tangent bundles, e.g. Poincaré spaces (spaces for which Poincaré duality is valid, essentially). Another reason is that it makes the construction of the Pontryagin–Thom map in Theorem 7.44 easier, notably because of the use of the tubular neighborhood theorem which privileges normal bundles.

Side remark 7.41. You may be wondering what happens if you take the Thom spectrum of the stable normal bundle $\nu : M \to BO$ of the manifold M; let us suppose M has no boundary. You get something very interesting: the *Spanier–Whitehead dual* of $\Sigma^\infty_+ M$. This phenomenon is called *Atiyah duality*; a textbook proof can be found in [**Rud98**, V.2.3], see also [**Ada74**, III.10]. Combining Atiyah duality with a Thom isomorphism similar to Theorem 7.28 but for Thom spectra, one gets a spectral proof of Poincaré duality. See [**BG99**] for a survey of these dualities and others.

Example 7.42. An important source of examples is given as follows. Let $\{G_n\}$ be a sequence of topological groups with maps $G_n \to G_{n+1}$ and morphisms $G_n \to O(n)$ making the obvious squares commute. After applying B, we get a map $\pi : BG \to BO$, where $G = \mathrm{colim}_n G_n$. We call a π-structure on a manifold M a *G-structure*.

(1) Let $G_n = O(n)$ with the obvious structure maps. Then the set of manifolds with O-structure is the same as the set of manifolds.
(2) Let $G_n = *$ with the obvious structure maps. Then a $*$-structure on a manifold M is equivalent to a framing of its stable normal bundle, i.e. an isomorphism between ν and the trivial bundle. This amounts to a stable framing of some normal bundle, i.e. a trivialization of a sum of it with a trivial bundle. We could have equivalently taken the stable tangent bundle, see [**DK01**, 8.13].

[17] We don't take maps to BF in this section. I'm not aware of a geometric Thom theorem for them as below. Also, note that this, or a variation of this, is often called a (B, f)-*structure* in the literature.

(3) Let $G_n = SO(n)$ with the obvious structure maps. Then an SO-structure on a manifold M is equivalent to an orientation of it.
(4) Include $U(n)$ into $O(2n)$, getting a map $BU \to BO$. A U-structure on a manifold M endows it with a *stable almost-complex structure*: an almost-complex structure in the stable normal bundle. This amounts to an almost-complex structure on the sum of a normal bundle with a trivial bundle, i.e. its fiber vector spaces have a complex action.
(5) We can also consider SU, getting *oriented stable almost-complex structures*. All of these are related via maps

$$* \longrightarrow SU \longrightarrow U \longrightarrow SO \longrightarrow O.$$

Remark 7.43. Recall the concept of a G-principal bundle for a topological group G: it's a map $E \to X$ where E has a right G-action and X is locally trivial with trivializations which are G-maps; the fibers are G-sets and the action is free and transitive, i.e. they are G-torsors.

There is a bijection between isomorphism classes of n-plane real vector bundles $\xi : E \to X$ over X and isomorphism classes of $GL_n(\mathbb{R})$-principal bundles p over X, which takes ξ to $E \times_{GL_n(\mathbb{R})} \mathbb{R}^n$ (change of fiber to $\mathbb{R}^n$) and takes p to its frame bundle (the fiber over x is the $GL_n(\mathbb{R})$-set of the fiber of ξ over x).

If X is a paracompact Hausdorff space, then the set of isomorphism classes of $O(n)$-principal bundles is also in bijection with the above.

If $G \subseteq O(n)$ is a topological subgroup, there is a similar theorem as above, using the concept of a fiber bundle with structure group, see [**DK01**, 4.4]. A fiber bundle with structure group $GL_n(\mathbb{R})$ and fiber $\mathbb{R}^n$ is precisely an n-plane real vector bundle.

You can identify fiber bundles with structure group G in many cases. For example, if $G = SO(n)$, then if the fibers are $\mathbb{R}^n$ these are oriented real n-plane vector bundles. If $G = U(n)$ and the fibers are $\mathbb{C}^n$, these are complex n-plane vector bundles; $SU(n)$ is the oriented variant.

If H is a topological subgroup of G, then a principal G-bundle over X is induced from a principal H-bundle by change of fiber if and only if the classifying map $X \to BG$ lifts to BH. This gives a geometrical interpretation of the homotopical approach taken in Definition 7.39.

We should now introduce the equivalence relation of G-cobordism between G-manifolds. We won't do that carefully; see [**Koc96**, 1.5.1], [**Rud98**, IV.7.25], [**Mil01**, 1.3]. We will only illustrate it with the example of SO, i.e. *oriented cobordism*. Here, manifolds are oriented, and an oriented cobordism between two oriented n-manifolds is an oriented $(n+1)$-manifold such that its boundary (with the boundary orientation) is the disjoint sum of them. This defines $\Omega_n^{SO}(*)$, and more generally, we can define the abelian group $\Omega_n^{SO}(X)$ for a space X.

Considering orientations like this eliminates the $\mathbb{Z}/2$ from Proposition 7.36. Indeed, if M is an oriented manifold and we take $M \times I$, now its boundary is not $M \sqcup M$ but $M \sqcup -M$, where $-M$ has the opposite orientation.

More generally, if $\{G_n\}$ is a system of groups as in Example 7.42, $G = \mathrm{colim}_n G_n$ and $BG \to BO$ its associated map, then we can define $\Omega_*^G(*)$, and more generally $\Omega_*^G(X)$. The fundamental theorem is the following:

Theorem 7.44 (Thom)**.** *Let $BG \to BO$ be as above. There is an isomorphism*

$$\Omega_*^G(X) \xrightarrow{\cong} M(BG \to BO \xrightarrow{J} BF)_*(X_+)$$

and thus G-cobordism is an unreduced homology theory. The map is the Pontryagin-Thom map. *In particular, $\Omega_*^G(*) \cong \pi_*(M(BG \to BO \xrightarrow{J} BF))$.*

We can apply this to the examples above. For example, we can apply it to the trivial groups. Then we get $\Omega_*^*(*) \cong \pi_*\mathbb{S}$: we reinterpret the stable homotopy groups of spheres as the abelian groups of stably framed manifolds up to stably framed cobordism (a manifold of one higher dimension which is stably framed and whose framing restricts to the original ones upon taking the boundary).

Thus, an element in $\pi_n(\mathbb{S})$ can be interpreted as a stably framed n-manifold. This point of view lead to some early computations of stable homotopy groups. This uses geometry to deduce something from stable homotopy theory; nowadays, the other direction is more common (cf. the recent Hill–Hopkins–Ravenel theorem on the Arf–Kervaire invariant [**HHR16**]).

CHAPTER 8

Introduction to brave new algebra

Classical algebra is the study of objects like abelian groups and rings. But even if one is interested only in these classical objects, it is a fruitful idea to embed these categories into larger ones were we can do new things. That is the idea of homological algebra/derived algebraic geometry: embed abelian groups into their chain complexes. There, you have notions of quasi-isomorphism, homotopy, projective resolution, etc., that allow us to define useful things like Tor, Ext, or the derived category.

Instead of using chain complexes, we could use spectra, and working over $\mathbb{S}$ instead of over $\mathbb{Z}$ gives us something more general. Many algebraic objects coming directly from homotopy theory just don't have chain representatives. Both approaches are linked, as we shall see.

We start with a categorical remark.

Remark 8.1. If $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal category, then we can form the category of commutative monoids $\mathrm{CMon}(\mathcal{C})$, which is also symmetric monoidal. We could also be looking at plain monoids instead of commutative ones.

If $A \in \mathrm{CMon}(\mathcal{C})$, then we can form the category $\mathrm{Mod}_A(\mathcal{C})$ of (left) A-modules. If $\mathcal{C}$ is closed (i.e. $- \otimes X$ is a left adjoint for all $X \in \mathcal{C}$), then there is a symmetric monoidal structure $(\mathrm{Mod}_A(\mathcal{C}), \otimes_A, A)$. The monoidal unit $\mathbf{1}$ is a commutative monoid, and its modules satisfy that $(\mathrm{Mod}_{\mathbf{1}}(\mathcal{C}), \otimes_{\mathbf{1}}, \mathbf{1}) \cong (\mathcal{C}, \otimes, \mathbf{1})$. The category $\mathrm{CMon}(\mathrm{Mod}_A(\mathcal{C}))$ is the category of commutative A-algebras.

Example 8.2.
(1) $\mathcal{C} = (\mathrm{Ab}, \otimes, \mathbb{Z})$. (Commutative) monoids here are (commutative) rings. The category of $\mathbb{Z}$-modules is equivalent to Ab.
(2) $\mathcal{C} = (\mathrm{Ch}_{\mathbb{Z}}, \otimes, \mathbb{Z})$, the category of unbounded chain complexes of abelian groups with its tensor product. Commutative monoids are called *commutative differential graded (dg) algebras*, also known as *cdga*s.
(3) $\mathcal{C} = (D(\mathbb{Z}), \otimes^L, \mathbb{Z})$, the derived category of $\mathbb{Z}$ with the derived tensor product. The canonical functor $\mathrm{Ch}_{\mathbb{Z}} \to D(\mathbb{Z})$ is symmetric monoidal, so a cdga gets sent to a commutative monoid in $D(\mathbb{Z})$, but a commutative monoid in $D(\mathbb{Z})$ is something generally weaker.
(4) $\mathcal{C} = (\mathrm{Top}, \times, *)$. A commutative monoid in here is exactly a topological commutative monoid.
(5) $\mathcal{C} = (\mathrm{Ho}(\mathrm{Top}), \times, *)$. A commutative monoid in here is exactly a homotopy commutative H-group.
(6) $\mathcal{C} = (\mathrm{Ho}(\mathrm{Sp}), \wedge, \mathbb{S})$. A commutative monoid in here is a *homotopy commutative ring spectrum*[1], and we get that $\mathrm{Ho}(\mathrm{Sp})$ is equivalent to the $\mathbb{S}$-modules. For coherence with the case of chain complexes, $\wedge$ should be denoted $\wedge^L$, and it should be called the *derived smash product*, as it's sometimes done.

[1] You may find it puzzling that this is called a "ring spectrum" and not a "monoid spectrum". The reason is Side remark 4.15: a connective spectrum is homotopically the same as an E_∞-group, which is the higher-homotopical version of an abelian group, so we already have one "additive" operation. It is also true that E_∞-ring spaces (undefined here) and connective E_∞-ring spectra are equivalent, up to homotopy. See [**May09a**].

Homotopy (commutative or not) ring spectra are useful for many things. For example, in [**Ada74**, II.10] they are used in a study of duality. The multiplicative structures pass to homotopy: the homotopy of a (homotopy commutative) ring spectrum is a (graded-commutative) graded ring and the homotopy of a module spectrum is a graded module over it.

However, these homotopy ring spectra are also for some purposes too little descriptive, just as saying that a loop space is an H-space is too little descriptive (cf. Side remark 4.15). Moreover, as we already know, the homotopy categories have lost too much information, so much so that homotopy (co)limits can't be formed in them.

So we would like to do with spectra something similar as with the chain complex examples above: not only consider monoids and modules in $D(\mathbb{Z})$ resp. Ho(Sp), but in $\mathrm{Ch}_{\mathbb{Z}}$ resp. Sp. The problem, as we know from Section 4, is that with our model of spectra we don't have such structure on Sp.

What is it that we want to achieve, really? Let us proceed more or less historically, as it makes the motivations clearer. Let $X \in \mathrm{Mon}(\mathrm{Ho}(\mathrm{Sp}))$, and suppose X is a cell Ω-spectrum. Then, for example, associativity means a diagram

$$\begin{array}{ccc} X \wedge X \wedge X & \xrightarrow{\mu \wedge \mathrm{id}} & X \wedge X \\ {\scriptstyle \mathrm{id} \wedge \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ X \wedge X & \xrightarrow[\mu]{} & X \end{array}$$

of spectra which is homotopy commutative, i.e. both maps $X \wedge X \wedge X \to X$ are homotopical. What we would like to record and axiomatize is similar to what we had in Side remark 2.10. For simplicity, suppose we are in spaces now. We can choose such homotopies for every space X in such a way that, when we take four elements of X, and multiply them to an element of X in the five different ways using the five homotopies induced from our choices before, this map $\partial K \times X^4 \to X$ can be filled to a map $K \times X^4 \to X$, where $K \subseteq \mathbb{R}^2$ is a solid regular pentagon. And this can be continued: we take five elements of X, etc.

The algebraic apparatus used to achieve this is an *operad*. An operad that encodes what we hand-wavingly specified above is an A_∞*-operad*. For example, one of the first A_∞-operads used in the setting of spectra was the *linear isometries operad*, see [**May77**, 1.1], [**May09b**, Section 2], [**EKMM97**, II.4]. An algebra over this operad in our category Sp would be an A_∞*-ring spectrum* (or prespectrum, as May and collaborators would say). The notation A_∞ stands for "associative up to an infinite sequence of higher homotopies".

The operad machinery also allows for modules. Just as an A_∞-ring spectrum improves a homotopy ring spectrum, a module over it[2] improves a homotopy module spectrum. A word of warning: different authors will use the words "ring spectrum" in different ways. They may mean a homotopy one, or an A_∞ one!

We may want to add "commutativity up to an infinite sequence of higher homotopies" – we replace A_∞ by E_∞, then. Here the E stands for "homotopy everything". Note that E_∞ is added *structure*, not merely a property (commutativity).

Remark 8.3. There are also A_n operads codifying homotopy associativity up to a sequence of higher homomotopies up to level n. More commonly in stable homotopy theory are the E_n operads: E_1 is A_∞, E_2 means A_∞ plus homotopy commutativity, E_3 means A_∞ plus coherent homotopy commutativity up to a single higher homotopy, etc.[3]

[2]We should say: "A_∞-module" but we will skip the A_∞ to make it simpler.

[3]For defining E_n, you could get away with only having A_{n+1}, but this is usually an unnecessary complication. See [**Wil69**].

This machinery, sketched above for spaces/spectra, works in many other situations. One can talk, for example, of E_∞-spaces, or of chain E_∞-algebra (chain complexes with an action of an E_∞-operad).

Remark 8.4. Some advantages of the highly structured ring/module spectra over the mere up-to-homotopy versions: we can endow them with a model structure/∞-category structure, so they have a homotopy theory of their own, and we can e.g. talk about homotopy (co)limits, which is always useful. Another improvement: if A is an E_∞-ring spectrum, then the category of A_∞-module spectra has a monoidal structure, which is not true if we're only up to homotopy. As a final example, high-brow example, we mention topological modular forms. The theorem of Goerss–Hopkins–Miller [**DFHH14**, 5.0.1] which identifies TMF as the global sections of a sheaf $\mathcal{O}^{\text{top}}$ of E_∞-ring spectra on the moduli stack of elliptic curves in the étale topology, would be impossible without highly-structured ring spectra. If you're intrigued by this, we recommend checking out [**Law09b**].[4]

Example 8.5.

1. The sphere spectrum $\mathbb{S}$ is an E_∞-ring spectrum. The multiplicative structure induced in $\pi_*\mathbb{S}$ was explored in the exercises.
2. The Eilenberg–Mac Lane spectrum of a (commutative) ring R gives an A_∞ (E_∞)-ring spectrum, and if M is a module over R, then HM is a module spectrum over HR.
3. The topological K-theory spectrum KU is an E_∞-ring spectrum, with multiplication coming from the tensor product of complex vector bundles.
4. The space BF is an E_∞-space. If X is also an E_∞-space and $f : X \to BF$ is an E_∞-map, then its Thom spectrum Mf is an E_∞-spectrum. Thus, MU is E_∞. In Example 7.33.(4), we mentioned that $H\mathbb{Z}$ and $H\mathbb{F}_2$ are equivalent to Thom spectra of maps $\Omega^2 S^3 \to BF$. These maps are only E_2, so the equivalence is merely of E_2-ring spectra, even though $H\mathbb{Z}$ and $H\mathbb{F}_2$ have full E_∞-structures.
5. The spectrum $K(1)$ at $p \neq 2$ which appeared in Example 7.23 is E_2, but not E_∞. The spectrum BP at the prime 2, mentioned in Example 7.33.(2), is an E_4-spectrum, but it is not E_{12} [**Law18**], answering after many years the open question of whether BP admits an E_∞-structure.

Remark 8.6. While definitely interesting and useful, A_∞-spaces do not give anything really new up to weak equivalence, since they are weak equivalent to topological monoids. This is a classical result from the beginnings of the theory, [**Sta63**], [**BV73**].

Similarly, every A_∞-algebra is quasi-equivalent to a dg-algebra. Keller calls it an "anti-minimal model" [**Kel01**]. I find it hard to navigate the literature on this, especially because the more algebraic sources prefer to work over a field. But I think an application of [**PS18**] gives the result over any commutative ring, and also encompasses the result for A_∞-spaces.

The analogous statement for E_∞ is not true. For example, if X is a topological space and R is a ring, then $C^*(X;R)$, the chain complex of singular chains with coefficients in R, is a chain E_∞-algebra, but it's not a cdga, nor quasi-isomorphic to one. There's a fundamental obstruction there, which accounts for the existence of the *Steenrod operations*. See Lurie: 18.917, Lecture 2, MO: Why does one think to Steenrod squares and powers?, [**MT68**, Pages 15-16]

[4]To entice you, here's the wonderful praise by Mike Hill on the MathSciNet review: "This paper provides a broad, clear introduction to the recent advances coming from the introduction of algebraic geometry machinery into homotopy theory. The author maintains an easy-going, conversational tone, peppering the discussion with amusing anecdotes and references. Though the breadth of the discussion is vast, the author does a very good job providing both just enough detail to explain the concepts and an extensive bibliography for the interested reader. Additionally, this paper has sufficient depth and enough examples to serve as a very good reference for beginning researchers in the field and for interested experts."

The general question of when is a coherent up-to-homotopy structure weakly equivalent to a strict structure is called the "rectification problem". Note it is a "model-dependent" question.

At any rate, considering the singular chains $C^*(X; R)$ together with its chain E_∞-algebra structure captures a lot of information of X. In fact, given hypotheses that in particular tame the π_1, it captures all the information of its homotopy type!

Theorem 8.7 (Mandell). *The homotopy category of finite type, nilpotent*[5] *spaces is equivalent to the homotopy category of chain E_∞-$\mathbb{Z}$-algebras. In particular, if X, Y are two such spaces, then X is weakly equivalent to Y if and only if $C^*(X; \mathbb{Z})$ is quasi-isomorphic to $C^*(Y; \mathbb{Z})$ as chain E_∞-$\mathbb{Z}$-algebras.*

Remark 8.8. The above is a fantastic theoretical result, that says that homotopy types can be modeled by a sufficiently sophisticated algebraic gadget, given some mild hypotheses. It improves on previous results of Quillen and Sullivan over $\mathbb{Q}$ (where, actually, one can model *rational spaces* by rational cdgas; see e.g. [**Hes07**], [**Ber12**] for surveys), and by Mandell himself for p-spaces.

This operadic approach to highly-structured ring spectra was the original one, and it underlies Luries approach to the subject, too [**Lur17**]. One can hide it all inside a black box if one gets a good smash product of spectra. Let us fix one of them: for example, let $\mathrm{Sp}^{\mathrm{EKMM}}$ denote the symmetric monoidal model category of spectra from [**EKMM97**], which is such that its homotopy category together with the induced smash product is monoidally equivalent to $(\mathrm{Ho}(\mathrm{Sp}), \wedge, \mathbb{S})$. Then the smash product there is engineered in such a way that monoids in $\mathrm{Sp}^{\mathrm{EKMM}}$ coincide with A_∞-ring spectra, commutative monoids coincide with E_∞-ring spectra, and modules coincide with A_∞-module spectra [**EKMM97**, II.4].[6] See [**Ric17**] for a nice survey on E_∞-ring spectra.

Remark 8.9. Here's a question you may ask: similarly as for spectra, can one build a symmetric monoidal category such that (commutative) monoids therein model A_∞- (E_∞-) spaces? Yes, you can. These are the $*$-*modules* introduced in Andrew Blumberg's Ph.D thesis, see also [**BCS10**] or [**ABG⁺14b**]. This mimicks a construction used in the foundations of EKMM spectra. See Remark 8.14 for an analog of this question for chain complexes.

Let us now compare this algebra with spectra with dg-algebra.

Theorem 8.10. *Let R be a ring. Consider the Eilenberg–Mac Lane spectrum HR as an A_∞-ring spectrum. The homotopy category of HR-module spectra is equivalent to the derived category of R.*

PROOF. This is a theorem of Robinson [**Rob87**]. It was improved in [**EKMM97**, IV.2.4] (see IV.2.2 there for a description of a functor associating a chain complex to a CW R-module) and improved to a 2-step zig-zag of Quillen equivalences of model categories in [**SS03**, 5.1.6, Appendix B]. It was also improved to an equivalence of ∞-categories by Lurie [**Lur17**, 7.1.1.16]. □

Remark 8.11. The theorem above is sometimes called the *stable Dold-Kan correspondence*, by analogy with the Dold-Kan correspondence which states an equivalence between the categories

[5]A space is of *finite type* if its homology is finitely generated in each degree; it is *nilpotent* if its fundamental group is nilpotent and acts nilpotently on the higher homotopy groups.

[6]The same is true for commutative monoids in symmetric and orthogonal spectra, provided one uses an adequate model structure, called the *positive* one. With the original ones, their homotopy theory is something else which has not been deemed interesting, I think. On the other hand, consider Γ-spaces: they model connective spectra, and I hear that Connes–Consani deem the commutative monoids in there, which do not model all connective E_∞-ring spectra [**Law09a**], as interesting in non-commutative geometry. As a final remark, see [**Lur17**, 4.5.4.7] for a theorem giving conditions on a monoidal model category such that commutative monoids in them model "the right thing" ∞-categorically.

of non-negative chain complexes and of simplicial abelian groups. There is an earlier precedent by Kan which makes the analogy clearer, see nLab:Stable Dold-Kan correspondence.

By contrast, there is no ring R, and indeed no abelian category $\mathcal{A}$ such that $\mathrm{Ho}(\mathrm{Sp}) \simeq D(\mathcal{A})$ by fairly elementary reasons; see [**Sch10**].

Remark 8.12. Combining this with an extension of scalars, we get a functor $\mathrm{Ho}(\mathrm{Sp}) \to D(\mathbb{Z})$. Indeed, for any map $R \to S$ of E_∞-ring spectra, we have an extension of scalars functor $S \wedge_R - : \mathrm{Mod}_R \to \mathrm{Mod}_S$, left adjoint to restriction of scalars. In particular, the Hurewicz map $\mathbb{S} \to H\mathbb{Z}$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Sp}^{\mathrm{EKMM}} \simeq \mathrm{Mod}_{\mathbb{S}} & \xrightarrow{H\mathbb{Z}\wedge -} & \mathrm{Mod}_{H\mathbb{Z}} \\ \downarrow \gamma & & \downarrow \gamma \\ \mathrm{Ho}(\mathrm{Sp}) & \xrightarrow[H\mathbb{Z}\wedge -]{} & \mathrm{Ho}(\mathrm{Mod}_{H\mathbb{Z}}) \simeq D(\mathbb{Z}). \end{array}$$

We thus have an "extension of scalars" functor $\mathrm{Ho}(\mathrm{Sp}) \to D(\mathbb{Z})$ along the Hurewicz map $\mathbb{S} \to H\mathbb{Z}$: the sphere spectrum $\mathbb{S}$ is the initial object among $A_\infty-$ and E_∞-ring spectra. This is closely related to the universal property mentioned in Side remark 6.12.

Theorem 8.10 takes care of the additive structure. What if we add a multiplication?

Theorem 8.13. *Let R be a commutative ring. The homotopy category of A_∞-HR-algebra spectra is equivalent to the homotopy category of differential graded R-algebras, which is equivalent to the homotopy category of chain A_∞-R-algebras.*

The homotopy category of E_∞-HR-algebra spectra is equivalent to the homotopy category of chain E_∞-R-algebras. These are equivalent to the homotopy category of R-cdgas if and only if R contains the rational numbers.

PROOF. The second statement of the first paragraph was mentioned in Remark 8.6. The first one is a theorem of Shipley [**Shi07**]. As for the second paragraph, this is a theorem of Shipley–Richter [**RS17**]. □

Under the second equivalence above, the chain E_∞-R-algebra from Theorem 8.7 corresponds to the E_∞-HR-algebra spectrum given by $F(X_+, HR)$.

Remark 8.14. Echoing Remark 8.9, we can now answer the following question. Does there exist a symmetric monoidal category whose commutative monoids are a good model for chain E_∞-R-algebras? Yes: that of HR-algebra spectra.[7] One can wonder if there exists a more "chain-complex flavored" strict model.

This is the entry-door to brave new algebra. A full development of the basics (and not only) in modern terms, adapting many old classical theorems to the homotopical setting, and proving many new things along the way, is in [**Lur17**].

To mention only two examples dear to the author of homotopical adaptations of classical theories, Hochschild homology (HH) of associative algebras has a counterpart for A_∞-ring spectra, called *topological Hochschild homology* (THH), and the theory of the cotangent complex / André–Quillen cohomology of commutative algebras also has one for E_∞-ring spectra (TAQ). Taking a classical ring R (like $\mathbb{Z}$) and identifying it with its Eilenberg–Mac Lane spectrum HR, we have a deeper base to work over, the sphere spectrum, and we can think of THH as being HH over $\mathbb{S}$ instead of over $\mathbb{Z}$, and similarly for TAQ. This is a very rich improvement which sheds light on classical algebra via this detour through the world of spectra, much like what

[7] More precisely: the commutative monoids in the symmetric monoidal category of HR-module symmetric spectra have a model structure which is Quillen equivalent to the model category of chain E_∞-R-algebras.

happens with derived algebra. And of course, there are brave new objects which do not come from classical algebra which are extremely interesting in themselves.

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