# Higher and iterated topological Hochschild homology

Notes for a talk at the 2015 European Talbot Workshop

Bruno Stonek

July 1, 2015

## Contents



## Notation

$\mathcal{T}$ is the category of based compactly generated weakly Hausdorff spaces.

$\mathcal{U}$ is the category of unbased compactly generated weakly Hausdorff spaces.

$\Delta$ is the *simplex category*, whose objects are non-empty finite ordinals $[n]$ and whose arrows are order-preserving maps.

**Fin** is the category whose objects are finite sets and whose arrows are functions.

$k$-$\mathbf{Mod}$, $k$-$\mathbf{Alg}$, $k$-$\mathcal{C}\mathbf{Alg}$ are the categories of $k$-modules, $k$-algebras and $k$-commutative algebras, respectively.

If $\mathcal{C}$ is a category, then $s\mathcal{C}$ is the category of simplicial $\mathcal{C}$-objects.

## 1 The algebraic case

Let $k$ be a commutative ground ring and $A$ be a commutative $k$-algebra. Consider the Hochschild homology of $A$, considered as a commutative simplicial $k$-algebra $HH_\bullet(A) : \Delta^{\mathrm{op}} \to k$-$\mathcal{C}\mathbf{Alg}$, i.e. $\big([n] \mapsto A^{\otimes(n+1)}\big)$ with the standard Hochschild faces and degeneracies.

Loday ([Lod89]) realized that this functor factors through the category of finite sets. More precisely, there is a functor

$$\Lambda_{(-)}(-) : \mathbf{Fin} \times k\text{-}\mathcal{C}\mathbf{Alg} \to k\text{-}\mathcal{C}\mathbf{Alg}$$

called the *(algebraic) Loday functor*, such that the following diagram commutes:

$$\begin{array}{ccc} \Delta^{\mathrm{op}} & \xrightarrow{HH_\bullet(A)} & k\text{-}\mathcal{C}\mathbf{Alg} \\ {\scriptstyle S^1}\downarrow & \nearrow_{\Lambda_{(-)}(A)} & \\ \mathbf{Fin} & & \end{array} \tag{1}$$

Here $S^1 : \Delta^{\mathrm{op}} \to \mathbf{Fin}$ is the standard simplicial circle. Explicitely, $S^1_q$ has $q+1$ simplices $\{x_0, \dots, x_q\}$, the face maps are $d_i(x_t) = \begin{cases} x_t & \text{if } t \le i \\ x_{t-1} & \text{if } t > i \end{cases}$ and the degeneracies are $s_j(x_t) = \begin{cases} x_t & \text{if } t \le j \\ x_{t+1} & \text{if } t > j \end{cases}$. Thus all of the simplices are degenerate except from one in degree zero and one in degree one.

By functoriality we can extend the Loday functor to a functor $s\mathbf{Fin} \times k\text{-}\mathcal{C}\mathbf{Alg} \to s(k\text{-}\mathcal{C}\mathbf{Alg})$ (and in fact, we can extend it to all of $s\mathbf{Set}$, using the fact that every set is the directed colimit of its finite subsets). Hence the commutativity of (1) can be expressed succinctly as

$$HH_\bullet(A) = \Lambda_{S^1}(A). \tag{2}$$

The definition of the Loday functor is as follows. If $X \in \mathbf{Fin}$ and $A \in k\text{-}\mathcal{C}\mathbf{Alg}$, define

$$\Lambda_X(A) = A^{\otimes |X|}. \tag{3}$$

If $(f, \phi) : (X, A) \to (Y, B)$, we associate to it the morphism

$$A^{\otimes |X|} \to B^{\otimes |Y|}, \qquad (a_x)_{x \in X} \mapsto \left( \prod_{x \in f^{-1}(y)} \phi(a_x) \right)_{y \in Y} \tag{4}$$

where if $f^{-1}(y)$ is empty then the product is interpreted to give the unit of $A$. Observe that we use the commutativity of $A$ for those products to make sense. [1]

The commutativity of (1) is now a simple verification.

This allows for the study of $\Lambda_X(A)$ for simplicial sets $X$ other than $S^1$. For example, higher algebraic Hochschild homology was introduced by Pirashvili ([Pir00]), who studied $\Lambda_{S^n}(A)$, which he called "Hochschild homology of order $n$" and which stabilizes to $\Gamma$-homology as introduced by Robinson & Whitehouse.

## 2 Tensored categories

**Definition 2.1.** A category $\mathcal{C}$ enriched over a closed symmetric monoidal category $\mathcal{V}$ is *tensored* if for every $V \in \mathcal{V}$ and $C \in \mathcal{C}$ there exists an object $V \otimes C \in \mathcal{C}$ such that there is an isomorphism in $\mathcal{V}$

$$\mathcal{C}(V \otimes C, C') \cong \mathcal{V}(V, \mathcal{C}(C, C')).$$

By general categorical arguments, we get a tensor bifunctor $\mathcal{V} \times \mathcal{C} \to \mathcal{C}$ and thus for every $C \in \mathcal{C}$, $V \in \mathcal{V}$ an adjunction

$$- \otimes C : \mathcal{V} \rightleftarrows \mathcal{C} : \mathcal{C}(C, -)\ . \tag{5}$$

[1] The formula has to be interpreted as follows. Choose for every $X \in \mathbf{Fin}$ an isomorphism with $[n]$ where $|X| = n+1$. This determines for every $f : X \to Y$ a map $f : [n] \to [m]$. Then the map (4) is $a_0 \otimes \cdots \otimes a_n \mapsto \prod_{i \in f^{-1}(0)} \phi(a_i) \otimes \cdots \otimes \prod_{i \in f^{-1}(m)} \phi(a_i)$. This is well defined and does not depend on the choice of isomorphisms of finite sets with their cardinals, by commutativity of $A$.

*Remark* 2.2. Denote by $\boxtimes$ the monoidal product in $\mathcal{V}$. Then if $U, V \in \mathcal{V}$, $C \in \mathcal{C}$, we have

$$(U \boxtimes V) \otimes C \cong U \otimes (V \otimes C).$$

This follows from uniqueness of adjoints, since both functors $- \otimes (V \otimes C), (- \boxtimes V) \otimes C : \mathcal{V} \to \mathcal{C}$ are left adjoints to the functor $\mathcal{V}(V, \mathcal{C}(C, -)) : \mathcal{C} \to \mathcal{V}$.

*Example* 2.3.

- If we take $\mathcal{V} = \mathcal{C}$, then the tensor is the monoidal product of $\mathcal{V}$.
- Every category $\mathcal{C}$ with copowers (i.e. with constant coproducts) is tensored over **Set** (which is cartesian closed, hence closed symmetric monoidal with $\otimes = \times$). The tensor $X \otimes C$ for $X \in \mathbf{Set}$ and $C \in \mathcal{C}$ is the copower $\coprod_{i \in X} C$.

  For example, $\mathcal{C} = k\text{-}\mathcal{C}\mathbf{Alg}$ is cocomplete. The coproduct is the tensor product, so for $X \in \mathbf{Set}$, $C \in k\text{-}\mathcal{C}\mathbf{Alg}$, we have $X \otimes C = C^{\otimes |X|}$.
- The category $s\mathbf{Set} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$ is cartesian closed pointwise. If $\mathcal{C}$ is a category with copowers then $s\mathcal{C}$ is tensored over $s\mathbf{Set}$: if $X \in s\mathbf{Set}$ and $C \in s(k\text{-}\mathcal{C}\mathbf{Alg})$, then $(X \otimes C)_n = \coprod_{i \in X(n)} C(n)$.

  For example, the category $s(k\text{-}\mathcal{C}\mathbf{Alg})$ is tensored over $s\mathbf{Set}$. Explicitely, if $X \in s\mathbf{Set}$ and $C \in s(k\text{-}\mathcal{C}\mathbf{Alg})$ then $(X \otimes C)_n = C(n)^{\otimes |X(n)|}$.

  Since the category $k\text{-}\mathcal{C}\mathbf{Alg}$ embeds in $s(k\text{-}\mathcal{C}\mathbf{Alg})$ as constant simplicial algebras, for $A \in k\text{-}\mathcal{C}\mathbf{Alg}$ and $X \in s\mathbf{Set}$ we get $X \otimes A \in s(k\text{-}\mathcal{C}\mathbf{Alg})$, $(X \otimes A)_n = X_n \otimes A$.
- We can thus express (3) for $X \in s\mathbf{Set}$ as:

  $$\Lambda_X(A) = X \otimes A, \tag{6}$$

  and in particular, we can express (2) as

  $$HH_\bullet(A) = S^1 \otimes A.$$

*Remark* 2.4. It follows from the Quillen adjunction $|-| : s\mathbf{Set} \leftrightarrow \mathbf{Top} : \mathrm{Sing}$ that giving an enrichment over $s\mathbf{Set}$ or over **Top** is equivalent. Same remark for the pointed versions. It also holds for the subcategories $\mathcal{U}, \mathcal{T}$ of nice spaces (see Notation at the beginning of the note).

## 3 Intermezzo: motivation and the main idea

**Main idea**: define a "topological Loday functor" which will make a sense out of the expression $THH(A) = \Lambda_{S^1}(A)$, for $A$ a commutative ring spectrum. Then define higher topological Hochschild homology as $\Lambda_{S^n}(A)$ and iterated topological Hochschild homology as $\Lambda_{T^n}(A)$ where $T^n$ is the $n$-torus.

Interest on $\Lambda_{S^n}(A)$ stems from the fact that there are stabilization maps $\pi_*(\Lambda_{S^n}(A)) \to \pi_{*+1}(\Lambda_{S^{n+1}}(A))$ whose colimit gives $TAQ_{*-1}(A)$, topological André-Quillen homology of $A$.

On the other hand, interest on $\Lambda_{T^n}(A)$ comes from its relation to algebraic $K$-theory: in words of [BCD10], $\Lambda_{T^n}(A)$ is proposed as "a computationally tractable cousin of $n$-fold iterated algebraic $K$-theory".

One has to be careful with the definitions. The commutative ring spectrum $THH(A)$, when suitably constructed, is endowed with an action of $S^1$. This allows for the study of topological cyclic homology, which is close to algebraic $K$-theory. We would like our Loday functor to respect this structure, i.e. the expression $\Lambda_{S^1}(A) = THH(A)$ to be an equivalence of $G$-equivariant commutative ring spectra for every finite subgroup $G$ of $S^1$. More generally, iterated $THH$ has a torus action, and we would like our Loday functor to respect it. This should then allow for definitions of iterated topological cyclic homology, closer to iterated algebraic $K$-theory, but we will not touch upon these matters.

The approach of [BCD10] is, as they say, "hands-on", and uses Segal's $\Gamma$-spaces as a foundation for spectra, and Bökstedt's original approach to $THH$. We choose to present the approach followed by [Sto11], which settles for orthogonal spectra. This has the advantage that the categorical tensor has the right equivariant homotopy type: we will define $\Lambda_X(A) := X \otimes A$, which wouldn't give an equivariantly correct model with the approach of [BCD10].

## 4 Orthogonal spectra and the topological Loday functor

Let us briefly recall the main definitions surrounding orthogonal spectra. The source for this section is [Sto11]: we refer the reader there for further details.

An *orthogonal spectrum* $X = \{X_n\}_{n\in\mathbb{N}}$ is a sequence of based spaces $X_n \in \mathcal{T}$ together with the following data:

1. For each $n \in \mathbb{N}$, an action of the orthogonal group $O_n$ on $X_n$,
2. For each $n \in \mathbb{N}$, a *structure map* $\sigma_n : X_n \wedge S^1 \to X_{n+1}$ such that for any $n, m \in \mathbb{N}$, the map

$$X_n \wedge S^m \to X_{n+m}$$

given by repeatedly applying the structure map adequately smashed with the identity, is $O_n \times O_m$-equivariant. Here $O_n \times O_m$ is naturally included in $O_{n+m}$ as block matrices.

A *morphism* of orthogonal spectra $f : X \to Y$ is a sequence of maps $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$ the following square commutes:

$$\begin{array}{ccc} X_n \wedge S^1 & \xrightarrow{f_n \wedge \mathrm{id}} & Y_n \wedge S^1 \\ \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma'_n} \\ X_{n+1} & \xrightarrow[f_{n+1}]{} & Y_{n+1} \end{array}$$

We denote by $\mathcal{S}p$ the category of orthogonal spectra. We have a sphere orthogonal spectrum $\mathbb{S}$, and $(\mathcal{S}p, \wedge, \mathbb{S})$ is a closed symmetric monoidal category. Commutative monoids in $\mathcal{S}p$ are called *commutative orthogonal ring spectra*. These form a category, whose objects $R$ we take as ground objects, and consider modules and algebras over these.

The category $\mathcal{S}p$ is enriched over the monoidal category $\mathcal{T}$, by topologizing the hom-set $\mathcal{S}p(X, Y)$ as the subset of the product $\prod_n \mathcal{T}(X_n, Y_n)$ of maps that make the above square commute. Moreover it is tensored over it: for an orthogonal spectrum $A$ and a based space $X$, we define the *tensor* $X \otimes A \in \mathcal{S}p$ as

$$(X \otimes A)_n = X \wedge A_n$$

with structure maps induced by the ones of $A$.

The category $\mathcal{S}p$ is also enriched and tensored over the monoidal (cartesian) category $\mathcal{U}$ of unbased spaces, by adjoining a base point (cf. [Sto11, (A.2.15)]). (Commutative) orthogonal ring spectra and their modules and algebras are also thus tensored over $\mathcal{U}$, by general categorical arguments (cf. [Sto11, (3.1.6)]).

**Definition 4.1.** Let $R$ be a commutative orthogonal ring spectrum. The *topological Loday functor*

$$\Lambda^R_{(-)}(-) : \mathcal{U} \times R\text{-}\mathcal{C}\mathbf{Alg} \to R\text{-}\mathcal{C}\mathbf{Alg}$$

is the categorical tensor:

$$\Lambda^R_{(-)}(-) = - \otimes -.$$

We will suppress the upper index when $R$ is the sphere spectrum.

We will use the following

*Properties* 4.2.

1. Let $X, Y \in \mathcal{U}$. Then

$$\Lambda^R_{X\times Y}(A) = \Lambda^R_X(\Lambda^R_Y(A)).$$

This follows from remark (2.2).

2. If $X \to Y$ is a weak equivalence of spaces, then the induced map $\Lambda_X A \to \Lambda_Y A$ is a weak equivalence.

3. The Loday functor preserves pushouts along cofibrations. More precisely, If $L \to X$ and $L \to K$ are maps between spaces and one of them is a cofibration, then there is a weak equivalence

$$\Lambda_{X\sqcup_L K} A \simeq \Lambda_X A \wedge_{\Lambda_L A} \Lambda_K A.$$

### 4.1 Equivariancy

Orthogonal spectra have the particularity that a naive approach to equivariancy actually captures the information at any orthogonal representation, thus justifying the following

**Definition 4.3.** Let $G$ be a topological group. A *$G$-orthogonal spectrum* is an orthogonal spectrum $X$ with a morphism of topological monoids $G \to \mathcal{S}p(X, X)$.

If $G$ is a topological group, $A$ is an $R$-commutative algebra and $X$ is a $G$-space then $\Lambda^R_X(A)$ is a $G$-orthogonal $R$-commutative algebra. Denote by $G\mathcal{U}$ the category of $G$-spaces, and by $G-R\text{-}\mathcal{C}\mathbf{Alg}$ the category of $G$-orthogonal $R$-commutative algebras. Then we get an equivariant Loday functor

$$\Lambda^R_{(-)}(-) : G\mathcal{U} \times R\text{-}\mathcal{C}\mathbf{Alg} \to G-R\text{-}\mathcal{C}\mathbf{Alg}.$$

We refer the reader to [Sto11, (2.1), (2.2.1), (3.2.7)] for details.

## 5 Higher and iterated $THH$

Topological Hochschild homology for orthogonal ring spectra is now defined in analogy with equation (2) for algebraic Hochschild homology:

**Definition 5.1.** Let $A$ be a commutative orthogonal ring spectrum. We define its *topological Hochschild homology*, which is an $S^1$-commutative orthogonal ring spectrum, as:

$$THH(A) := \Lambda_{S^1}(A).$$

We also define

$$\Lambda_{T^n}(A) \;=:\; \textit{the } n\textit{-th iterated topological Hochschild homology of } A$$

and

$$\Lambda_{S^n}(A) \;=:\; \textit{the } n\textit{-th higher topological Hochschild homology of } A.$$

*Remark* 5.2. Since $T^n = (S^1)^n$, (4.2.1) implies that iterated $THH$ deserves its name:

$$\Lambda_{T^n}(A) = \overbrace{THH(\ldots(THH}^{n \text{ times}}(A))).$$

Higher $THH$ has hitherto been studied further than iterated $THH$ (see e.g. [DLR15], [Sch11]). We choose to describe the case of $A = H\mathbb{F}_p$ as studied in [Vee14], for it exhibits interesting computations in both higher and iterated $THH$.

## 6 The case of $H\mathbb{F}_p$

In [Vee14], Veen calculates the homotopy groups of higher and iterated $THH$ of the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$ in a certain range. Let us review his results (but see also the last sentence at the end of (6.1)).

### 6.1 Higher $THH$ of $H\mathbb{F}_p$

The main tool Veen uses to compute $\pi_*(\Lambda_{S^n} H\mathbb{F}_p)$ is the bar spectral sequence, which is the spectral sequence arising from the skeletal filtration of a simplicial spectrum [EKMM97, X.2.9], applied to the bar construction [EKMM97, IV.7.7]. The particular incarnation we will be handling is the following: if $A$ is a bounded-below orthogonal ring spectrum, $M$ is a right $A$-module and $N$ is a left $A$-module, there is a strongly convergent, right half-plane, homological spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*(A)}_{s,t}(\pi_* M, \pi_* N) \Rightarrow \pi_{s+t}(B(M, A, N)) = \pi_{s+t}(M \wedge^L_A N). \tag{7}$$

Recall that Bökstedt proved that $THH_*(H\mathbb{F}_p) = \pi_*(\Lambda_{S^1} H\mathbb{F}_p) = \mathbb{F}_p[\mu],\ |\mu| = 2$. The result we will get will be expressed by taking iterated Tors starting with this algebra.

**Definition 6.1.** We denote by $B_n$ be the $\mathbb{F}_p$-Hopf algebra defined recursively as follows:

$$B_1 = \mathbb{F}_p[\mu], \quad |\mu| = 2,$$
$$B_n = \mathrm{Tor}^{B_{n-1}}_*(\mathbb{F}_p, \mathbb{F}_p).$$

Cartan [Car54] determined an explicit presentation of such algebras. A more recent account is Proposition 7.4 in [Vee14]. For example, $B_2 = \mathrm{Tor}^{\mathbb{F}_p[\mu]}_*(\mathbb{F}_p, \mathbb{F}_p) = E_{\mathbb{F}_p}(\beta)$ with $|\beta| = 3$, where $E$ denotes an exterior algebra.

**Theorem 6.2.** *If $p$ is an odd prime and $n \leq 2p+2$, then there is an $\mathbb{F}_p$-Hopf algebra isomorphism* [Vee14, (7.6)], [BLP+14, (4.4)]*:*

$$\pi_*(\Lambda_{S^n} H\mathbb{F}_p) \cong B_n. \tag{8}$$

*In the case $p = 2$, the isomorphism is also true for $n = 2$* [Vee13, (2.3.1)], *and a similar argument shows that it is true for $n = 3$* [BLP+14, (5.1)].

*Comment on the proof:* The idea is to inductively use the bar spectral sequence (7). If $R$ is a commutative orthogonal ring spectrum and $X \in \mathcal{U}$, then since $\Sigma X$ is the pushout of $CX \leftarrow X \rightarrow CX$, by 4.2.2 and 4.2.3 we get

$$\Lambda_{\Sigma X} R \simeq R \wedge_{\Lambda_X R} R \simeq B(R, \Lambda_X R, R)$$

and thus the bar spectral spectral sequence computing $\pi_*(B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, H\mathbb{F}_p))$ takes the form

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*(\Lambda_{S^{n-1}} H\mathbb{F}_p)}_{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{s+t}(\Lambda_{S^n} H\mathbb{F}_p). \tag{9}$$

This is an $\mathbb{F}_p$-Hopf algebra spectral sequence, and this extra structure is very valuable in the computations. The base case $\pi_*(\Lambda_{S^1} H\mathbb{F}_p) = B_1$ goes back to Bökstedt. Thus, the $E^2$-term in (9) for $n = 2$ is $B_2$.

Now the isomorphism (8) for any $p$ and for $n = 2$ follows easily, for the $E^2$-term in (9) is $E_{\mathbb{F}_p}(\beta)$ which is concentrated in bidegrees $(0, 0)$ and $(1, 2)$, and this immediately gives that there are no non-trivial differentials and no non-trivial extensions. Thus we get that the $E^2$-term in (9) for $n = 3$ is $B_3$. In this $n = 3$ case, we can also similarly conclude that the spectral sequence collapses at $E^2$ and that there are not non-trivial extensions, after identifying $B_3$ as a divided power algebra.

For $p$ odd and $n \leq 2p + 2$, once again the spectral sequence collapses at the $E^2$-page and there are no non-trivial extensions. For $n \leq 2p$ this was proven by Veen, and the bound was pushed to $n \leq 2p + 2$ in [BLP+14]. The arguments depend on more careful analysis of the spectral sequence. □

I should now say that, actually, the isomorphism (8) holds for all $n$ and all $p$. This is proven in [DLR15], although in its current preprint version this result is not explicitely stated, though it follows from their calculations.

### 6.2 Iterated $THH$ of $H\mathbb{F}_p$

**Theorem 6.3.** [Vee14, (1.1)] *Let $p$ be an odd prime and $1 \leq n \leq p$ if $p \geq 5$, and $1 \leq n \leq 2$ if $p = 3$. Then there is an $\mathbb{F}_p$-algebra isomorphism*

$$\pi_*(\Lambda_{T^n} H\mathbb{F}_p) \cong \bigotimes_{k=1}^{n} B_k^{\otimes \binom{n}{k}} \tag{10}$$

*where the $B$-terms on the right correspond to definition 6.1.*

*Comment on the proof:* We use another incarnation of our bar spectral sequence (7). Give $T^n$ the structure of a product CW-complex, where $S^1$ has one 0-cell and one 1-cell. Then $T^n$ has one top $n$-cell: we have a pushout square

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & T^n_{n-1} \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & T^n, \end{array}$$

and therefore by 4.2.2 and 4.2.3 we get

$$\begin{aligned} \Lambda_{T^n} H\mathbb{F}_p &\simeq H\mathbb{F}_p \wedge_{\Lambda_{S^{n-1}} H\mathbb{F}_p} \Lambda_{T^n_{n-1}} H\mathbb{F}_p \\ &\simeq B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^n_{n-1}} H\mathbb{F}_p). \end{aligned}$$

The bar spectral sequence (7) for this last term thus takes the form

$$E^2_{s,t} = \mathrm{Tor}_{s,t}^{\pi_*(\Lambda_{S^{n-1}} H\mathbb{F}_p)}(\mathbb{F}_p, \pi_*(\Lambda_{T^n_{n-1}} H\mathbb{F}_p)) \Rightarrow \pi_{s+t}(\Lambda_{T^n}(H\mathbb{F}_p)). \tag{11}$$

To identify this $E^2$-term, Veen uses the Bökstedt spectral sequence computing the homology of $THH(\Lambda_{T^{n-1}} H\mathbb{F}_p) = \Lambda_{T^n} H\mathbb{F}_p$:

$$HH_*(H_*(\Lambda_{T^{n-1}} H\mathbb{F}_p; \mathbb{F}_p)) \Rightarrow H_*(\Lambda_{T^n} H\mathbb{F}_p; \mathbb{F}_p).$$

An long and intricate analysis of (11) ensues. Compare the final description with the usual cell decomposition of $T^n$. □

Ausoni and Dundas have work in progress in the direction of extending the isomorphism (10) to all $n$ and $p$.

### 6.3 Periodic elements

In this more hand-wavy paragraph I will outline the connection of Veen's work to Ausoni-Rognes' redshift conjecture. My understanding of these matters is severely sketchy, so a big *caveat lector* is in order.

Recall that the algebraic $K$-theory of $H\mathbb{F}_p$ is in chromatic level 0: indeed, we have

$$K(H\mathbb{F}_p)^\wedge_p \simeq H\mathbb{Z}_p$$

which has non-trivial $K(0)$-homology and trivial $K(m)$-homology for $m \geq 1$; here $K(m)$ denotes the $m$-th Morava $K$-theory at the prime $p$.

The $K$-theory of $H\mathbb{Z}_p$, completed at $p$, is at chromatic level 1: it is related to topological $K$-theory. As an incarnation of the redshift conjecture, we wonder whether the $n$-fold algebraic $K$-theory of $H\mathbb{F}_p$ completed at $p$ is at chromatic level $n-1$, i.e. if its $K(m)$-homology is non-trivial up to $n-1$ and is trivial from $n$ onwards. A first approach to this is to consider $n$-fold $THH$. But this does not work as $THH$ does not increase chromatic levels: for $A$ commutative, $THH(A)$ is an $A$-module, and thus $K(n)_*(A) = 0$ implies $K(n)_*(THH(A)) = 0$ by looking at the unit map $A \to THH(A)$.

Recall from section (4.1) that $\Lambda_{T^n} H\mathbb{F}_p$ has a $T^n$-action. The iterated Dennis trace map $K^{(n)} H\mathbb{F}_p \to \Lambda_{T^n} H\mathbb{F}_p$ factors through the homotopy fixed points $(\Lambda_{T^n} H\mathbb{F}_p)^{hT^n}$, which are a higher analogue of negative cyclic homology $HC^-(\mathbb{F}_p)$ (see [Hoy15] for a proof that $HC^-(A) = HH(A)^{S^1}$).

We wonder if $(\Lambda_{T^n} H\mathbb{F}_p)^{hT^n}$ is at chromatic level $n-1$: is $v_{n-1}$ in

$$K(n-1)_*((\Lambda_{T^n} H\mathbb{F}_p)^{hT^n}) \tag{12}$$

non-zero? Connective Morava $K$-theory at the prime $p$, denoted $k(n-1)$, can detect $v_{n-1}$ periodicity, and allows for finer analysis than the periodic theory. Indeed, $v_{n-1}$ in (12) is non-zero if and only if all powers of $v_{n-1}$ in

$$k(n-1)_*((\Lambda_{T^n} H\mathbb{F}_p)^{hT^n}) \tag{13}$$

are non-zero. Veen [Vee14, (10.7)] proves the first step in this direction: he proves that $v_{n-1}$ is non-zero in (13). In particular, since there is a canonical map $K^{(n)} H\mathbb{F}_p \to (\Lambda_{T^n} H\mathbb{F}_p)^{hT^n}$, then $v_{n-1}$ in

$$k(n-1)_*(K^{(n)} H\mathbb{F}_p)$$

is non-zero too.

# References


[BCD10] Morten Brun, Gunnar Carlsson, and Bjørn Ian Dundas, *Covering homology*, Adv. Math. **225** (2010), no. 6, 3166–3213. MR 2729005 (2012c:19003)

[BLP+14] Irina Bobkova, Ayelet Lindenstrauss, Kate Poirier, Birgit Richter, and Inna Zakharevich, *On the higher topological hochschild homology of $\mathbb{F}_p$ and commutative $\mathbb{F}_p$-group algebras*, ArXiv preprint, 1312.6378:v2 (2014).

[Car54] H Cartan, *Détermination des algèbres $H_*(\pi, n; Z)$*, Séminaire Henri Cartan **7** (1954), no. 1, 1–24.

[DLR15] Bjørn Ian Dundas, Ayelet Lindenstrauss, and Birgit Richter, *On higher topological Hochschild homology of rings of integers*, ArXiv e-prints, 1502.02504 (2015).

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR 1417719 (97h:55006)

[Hoy15] Marc Hoyois, *The fixed points of the circle action on Hochschild homology*, ArXiv e-prints, 1506.07123 (2015).

[Lod89] Jean-Louis Loday, *Opérations sur l'homologie cyclique des algèbres commutatives*, Invent. Math. **96** (1989), no. 1, 205–230. MR 981743 (89m:18017)

[Pir00] Teimuraz Pirashvili, *Hodge decomposition for higher order Hochschild homology*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 2, 151–179. MR 1755114 (2001e:19006)

[Rie14] Emily Riehl, *Categorical homotopy theory*, New Mathematical Monographs, vol. 24, Cambridge University Press, Cambridge, 2014. MR 3221774

[Sch11] Christian Schlichtkrull, *Higher topological Hochschild homology of Thom spectra*, J. Topol. **4** (2011), no. 1, 161–189. MR 2783381 (2012b:55006)

[Sto11] Martin Stolz, *Equivariant structure on smash powers of commutative ring spectra*, PhD Thesis, University of Bergen (2011).

[Vee13] Torleif Veen, *Detecting periodic elements in higher topological Hochschild homology*, PhD Thesis, University of Bergen (2013).

[Vee14] ______, *Detecting periodic elements in higher topological Hochschild homology*, ArXiv e-prints, 1312.5699v2 (2014).