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Sur l'homologie de Hochschild topologique supérieure et itérée
On higher and iterated topological Hochschild homology

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Abstract

We work with topological Hochschild homology (THH) and two different ways of iterating it. If A is an augmented commutative R-algebra, where R is an E_∞-ring spectrum, then the n-th higher reduced topological Hochschild homology is the n-th iteration of $THH^R(A, R)$. We also consider the n-th iteration of $THH^{\mathbb{S}}(R)$, denoted $THH^n(R)$.

In the first part, we provide a detailed account of the foundations, namely symmetric monoidal categories $\mathcal{V}$ and bar constructions in $\mathcal{V}$. We prove that if a cosimplicial object $D^\bullet$ is induced from the canonical cosimplicial simplicial set, then the geometric realization functor $| - |_{D^\bullet}$ is monoidal, and this monoidality is preserved by left adjoint monoidal functors via a *monoidal* natural isomorphism.

We establish the existence of a graded multiplication in higher reduced topological Hochschild homology in a cartesian setting.

Let KU denote the periodic complex K-theory spectrum. We prove $THH^n(KU) \simeq KU\left[\prod_{i=1}^n K(\mathbb{Z}, i+2)^{\times\binom{n}{i}}\right]$ as commutative KU-algebras. We describe its augmentation ideal as a non-unital commutative $KU_{\mathbb{Q}}$-algebra. It is $\Sigma KU_{\mathbb{Q}}$ with the trivial multiplication when $n = 1$. Finally, we prove that $\Sigma Y \otimes KU \simeq F(\Sigma Y \wedge KU_{\mathbb{Q}})$, for Y a based CW-complex, where F is the free commutative KU-algebra on a KU-module functor.

Keywords: Homotopy theory, Category theory, Topological Hochschild homology, Structured ring spectra, Complex K-theory

Résumé

On étudie l'homologie de Hochschild topologique (THH) et deux façons différentes de l'itérer. Si A est une R-algèbre commutative augmentée, où R est un spectre en anneaux E_∞, alors son n-ième homologie de Hochschild topologique réduite est l'n-ième itération de $THH^R(A, R)$. On considère aussi l'n-ième itération de $THH^{\mathbb{S}}(R)$, notée $THH^n(R)$.

Dans la première partie, on fait un rapport détaillé des fondements, notamment des catégories monoïdales symétriques $\mathcal{V}$ et des constructions bar dans $\mathcal{V}$. On démontre que si un objet cosimplicial $D^\bullet$ est induit par l'ensemble simplicial cosimplicial canonique, alors la réalisation géométrique $|-|_{D^\bullet}$ est monoïdale, et cette monoïdalité est preservée par les foncteurs monoïdaux adjoints à gauche à travers un isomorphisme naturel *monoïdal*.

On établit l'éxistence d'une multiplication graduée dans l'homologie de Hochschild topologique supérieure réduite dans un cadre cartésien.

Soit KU le spectre en anneaux de la K-théorie complexe périodique. On démontre que $THH^n(KU) \simeq KU\left[\prod_{i=1}^n K(\mathbb{Z}, i+2)^{\times\binom{n}{i}}\right]$ comme KU-algèbres commutatives. On décrit son idéal d'augmentation comme une $KU_{\mathbb{Q}}$-algèbre commutative non-unitaire. Il s'agit de $\Sigma KU_{\mathbb{Q}}$ avec une multiplication triviale quand $n = 1$. Finalement, on démontre que $\Sigma Y \otimes KU \simeq F(\Sigma Y \wedge KU_{\mathbb{Q}})$ pour un complexe CW pointé Y, où F est le foncteur de KU-algèbre commutative libre sur un KU-module.

Mots-clés: Théorie de l'homotopie, Théorie des catégories, Homologie de Hochschild topologique, Spectres en anneaux commutatifs, K-théorie complexe

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Contents

Introduction

The central object of study of this dissertation is topological Hochschild homology (THH). It was introduced with this name in an unpublished manuscript by Bökstedt [**Bök85**], though it already appeared in a different guise in the work of Breen [**Bre78**].

Perhaps the easiest way to introduce THH to somebody familiar with ordinary Hochschild homology is: instead of working in the symmetric monoidal category of modules over a commutative ring, we will carry out the analogous constructions in a symmetric monoidal category of modules over a strictly associative and commutative ring spectrum in the sense of stable homotopy theory.

One should be aware that this was not how it was historically introduced. When Bökstedt was starting his work on THH, there was no such symmetric monoidal category. In modern words, one might say that algebraic topologists were looking for an adequate symmetric monoidal stable model category $\mathcal{M}$ such that its homotopy category $\mathrm{Ho}(\mathcal{M})$ would be equivalent, as a symmetric monoidal triangulated category, to the stable homotopy category of spectra $\mathcal{SHC}$ with the derived smash product and the standard triangulated structure. Lewis [**Lew91**] proved that there could be *no* such $\mathcal{M}$ satisfying a list of five reasonable axioms. However, not long after that, there were not one but several options for an $\mathcal{M}$ that would drop one of Lewis' axioms, like the S-modules of [**EKMM97**] and the symmetric spectra of [**HSS00**].

Bökstedt did not have these tools at hand, so he introduced THH in a different way (see [**Bök85**], or [**DGM13**] for a more modern account). As soon as the modern categories of spectra were available, the definition mimicking ordinary Hochschild homology was possible [**EKMM97**, Chapter IX], [**Shi00**]. See also [**MS93**, Section 3], where the construction is carried out in this way, but the discovery of modern spectra not having yet been made, the authors were forced to state: "*we must now admit that there is no known category of spectra in which strictly associative ring spectra can even exist*". Luckily, it was only a year until the appeareance of S-modules.

One reason for the importance of THH is the relation to algebraic K-theory. If R is a (discrete) ring, then the trace map $K(R) \to HH(R)$ factors through the topological Hochschild homology of the Eilenberg-Mac Lane ring spectrum of R. Moreover, algebraic K-theory and the trace map $K(A) \to THH(A)$ exist for any ring spectrum A. Out of topological Hochschild homology one can build topological cyclic homology, which has close ties to algebraic K-theory; see [**DGM13**]. We might thus see THH as a more easily approachable stepping stone on the way to the more fundamental algebraic K-theory.

Soon after the introduction of good symmetric monoidal categories of spectra, it was realized [**MSV97**] that the topological Hochschild homology of a *commutative* ring spectrum R (i.e. strictly associative and commutative) could be expressed as a tensor: the symmetric monoidal category of commutative ring spectra is tensored (in the sense of [**Kel05**, 3.7]) over the category of unbased spaces, and $THH(R)$ coincides exactly with $S^1 \otimes R$, where $\otimes$ denotes this tensor.

This begets the question: what can be said of $X \otimes R$ for other spaces X? In particular, what happens for spheres and tori? The author is aware of no earlier exploration of this question in this context than [**BCD10**], [**CDD11**] and [**Vee13**]; see also [**BLP**$^+$**15**]. However, in the context of ordinary Hochschild homology, already Pirashvili [**Pir00**] had begun the study of what he called *higher Hochschild homology*, which is the analog in the algebraic setting of choosing higher spheres.

For $X = T^n$ an n-dimensional torus, $T^n \otimes R$ is *n-fold iterated* topological Hochschild homology: indeed, $THH(R)$ for R commutative carries the structure of a commutative ring spectrum, so we can apply THH to it, and this construction makes sense. We can thus denote $T^n \otimes R$ by $THH^n(R)$. See [**CDD11**]: they propose $THH^n(R)$ as "a computationally tractable cousin of n-fold iterated algebraic K-theory".

All of these constructions can be carried over a different base ring spectrum. The initial ring spectrum is $\mathbb{S}$, the sphere spectrum, so that ring spectra are $\mathbb{S}$-algebras; we can just as well consider algebras over other commutative ring spectra. So if A is an R-algebra, then $THH^R(A)$ denotes this relative THH.

Just as for ordinary Hochschild homology, if A is an R-algebra and M is an A-bimodule, there is also a relative THH with coefficients in M: $THH^R(A, M)$. When A is an *augmented* R-algebra, meaning that it has an R-algebra map down to R, then we can consider $THH^R(A, R)$, the *reduced* THH^R of A.

The category of augmented R-algebras is tensored over *pointed* spaces: denote by $\odot$ its tensor. It turns out that $THH^R(A, R)$ coincides with $S^1 \odot R$, where we consider S^1 as a pointed space [**Kuh04**]. In this augmented/pointed context, the n-fold iteration of $THH^R(A, R)$ coincides with $S^n \odot R$.

The main results of this dissertation concern both kinds of iteration of THH, the augmented and the non-augmented one.

First, we need to take a close look at the construction of topological Hochschild homology itself: both of $THH^R(A, R)$ and of $THH^R(A)$. They are formed in a two-step process. The first step is different according to which version of THH we want to obtain: it will be some kind of *simplicial bar construction*, a procedure by which one obtains a simplicial R-module. Then comes the second step, which is geometric realization.

In fact, these simplicial bar constructions make sense in any symmetric monoidal category $\mathcal{V}$. Denote by **Sp** some symmetric monoidal category of spectra. A commutative ring spectrum R is a commutative monoid in **Sp**, an R-module is a module over this monoid, and an R-algebra is a monoid in this category of R-modules. These are all notions from monoidal category theory, which we expound in **Chapter 1**.

The R-module $THH^R(A, R)$ will be the geometric realization of the simplicial module $B_\bullet(A) = B_\bullet(R, A, R)$, the former being called *simplicial (reduced) bar construction* and the latter being an instance of the *simplicial two-sided bar construction*. On the other hand, $THH^R(A)$ is the geometric realization of a *simplicial cyclic bar construction*. These constructions and their relations are investigated in **Chapter 2** in the generality of symmetric monoidal categories. We will pay special attention to the multiplicative structures present in these objects. Note the naming convention: we drop the adjective "simplicial" whenever the constructions have been geometrically realized.

Chapter 3 deals with geometric realizations. This is the machinery necessary to realize a simplicial object in a symmetric monoidal category $\mathcal{V}$ as an object of $\mathcal{V}$ itself. This is not hard to obtain abstractly: if we are given a chosen cosimplicial object D in $\mathcal{V}$, then given $X \in s\mathcal{V}$ (the s stands for "simplicial objects"), we could define a geometric realization as a tensor product of functors: $|X| = X \otimes_{\mathbb{\Delta}} D$.

However, we want $| - |$ to respect the multiplicative structure that the simplicial bar constructions will have: in particular, because we want to iterate the bar construction. So if we start from an R-algebra A, we want the geometric realization of a simplicial bar construction on A to be an R-algebra again, not merely an R-module.

For this reason, and because all the examples that we want to consider follow this pattern, we consider only cosimplicial objects D which are induced from the canonical cosimplicial simplicial set $\Delta^\bullet$, namely the Yoneda embedding. By this we mean that we suppose given a symmetric monoidal functor $F : s\mathbf{Set} \to \mathcal{V}$, and we take $D = F\Delta^\bullet$. The geometric realization functor $| - | : s\mathcal{V} \to \mathcal{V}$ defined by such a cosimplicial object does indeed take a simplicial commutative monoid into a commutative monoid: this is **Theorem 3.9**. Note that this theorem does not involve bar constructions: indeed, we believe that the results in this chapter have an intrinsic interest which lies beyond their standing as technical tools for their application to bar constructions in the later chapters.

The next important result in this chapter is the following. Let $G : \mathcal{V} \to \mathcal{W}$ be a strong symmetric monoidal functor which is a left adjoint. Then **Theorem 3.11** provides us with a *monoidal* natural isomorphism $|G - | \Rightarrow G| - |$ of functors $s\mathcal{V} \to \mathcal{W}$. The fact that this natural isomorphism is monoidal is doubly important. First, it is an interesting fact *per se*, and it has applications. As a particular instance of this theorem, we obtain for example that the isomorphism of spectra $\Sigma^\infty_+|X_\bullet| \cong |\Sigma^\infty_+ X_\bullet|$ where X is a simplicial topological space, is actually monoidal (Example 5.1). Second, the monoidality of this natural isomorphism implies that it begets a natural isomorphism $|GA| \cong G|A|$ of *augmented commutative monoids*, if A is a simplicial augmented commutative monoid in $\mathcal{V}$, and this is crucial to our desires: it will allow us to compare bar constructions in different categories.

In **Chapter 4** we consider BA, the (reduced) bar construction of an augmented commutative monoid A in the symmetric monoidal category $\mathcal{V}$. We iterate it to get a graded augmented commutative monoid $(B^nA)_{n\geq 0}$, i.e. a sequence of augmented commutative monoids. Our quest is now to find a graded multiplication

$$B^nA \otimes B^mA \to B^{n+m}A. \tag{0.1}$$

To achieve this, we will need the monoid A itself to have an additional multiplication. We need to specialize the context: instead of considering general symmetric monoidal categories, we will need to consider *cartesian* ones, i.e. those where the tensor product is a categorical product. Such a category allows for *ring objects*. Note that a ring object is in particular an augmented commutative monoid with a trivial augmentation. It should also be noted that from any symmetric monoidal category we can define an interesting cartesian category, the one formed by its cocommutative comonoids.

If A is a ring object in a cartesian category $\mathcal{C}$, then the iterated bar constructions $B^*A = (B^nA)_{n\geq 0}$ get a graded multiplication: B^*A is a graded ring object. This is the content of **Theorem 4.1**. Moreover, if $G : \mathcal{C} \to \mathcal{D}$ is a left adjoint, cartesian functor between cartesian categories, then we get a natural isomorphism of graded ring objects $B^*GA \cong GB^*A$.

Note that so far everything has taken place in a general symmetric monoidal (or cartesian) category. In **Chapter 5** we unveil some examples, with focus on the reduced bar construction and applications of the graded multiplication. When we take the category of topological spaces as our cartesian category, then given a topological abelian group A, BA is the bar construction introduced by Milgram [**Mil67**]. When A is discrete, B^nA serves as a model for an Eilenberg-Mac Lane space $K(A, n)$. Starting from a ring R, the graded multiplication (0.1), which here takes the form $K(R, n)\times K(R, m) \to K(R, n+m)$, is the one found by Ravenel and Wilson in [**RW80**], which gives the cup product in singular cohomology with coefficients in R. We can also run this machine in the category of simplicial sets. If A is a simplicial abelian group, then B^nA gives a simplicial model for a $K(A,n)$; this is well-known. But we also get that starting from a simplicial ring R, B^*R is a graded simplicial ring, which under the geometric realization functor corresponds to the topological construction of Ravenel and Wilson. This was to be expected.

We can also work in some symmetric monoidal category of spectra. Fix R a commutative ring spectrum. Then if A is an augmented commutative R-algebra, BA is the topological Hochschild homology $THH^R(A,R)$ over R relative to R. The iterations B^nA model the *higher* topological Hochschild homology $THH^{R,[n]}(A,R) = S^n \odot A$, where $\odot$ denotes the tensoring over pointed topological spaces. Now, if A is a ring object in cocommutative R-coalgebras, then we get a graded multiplication in higher THH (5.16):

$$THH^{R,[n]}(A,R) \wedge_R THH^{R,[m]}(A,R) \to THH^{R,[n+m]}(A,R).$$

Moreover, if A is of the form $R[S]$ where S is a ring, then

$$THH^{R,[*]}(R[S],R) \cong R[K(S,*)],$$

a natural isomorphism of graded ring objects in cocommutative R-coalgebras (5.17). Here $R[-]$ denotes $R \wedge_S \Sigma^\infty_+$.

In **Chapter 6** we shift our attention from reduced relative THH to absolute THH, i.e. over the sphere spectrum. Here we deal with a very concrete example: KU, the commutative ring spectrum of periodic complex topological K-theory. We will use the theory of $\mathbb{S}$-algebras of [**EKMM97**] throughout.

Previously, McClure and Staffeldt [**MS93**, Theorem 8.1] showed that $THH(L) \simeq L \vee \Sigma L_{\mathbb{Q}}$ as spectra, where L is the p-adic completion of the Adams summand of KU for a given odd prime p; the result was extended to $p = 2$ in [**AHL10**, 2.3]. In Corollary 7.9 of [**AHL10**], the authors show that $THH(KO) \simeq KO \vee \Sigma KO_{\mathbb{Q}}$ as KO-modules. Here KO denotes the commutative ring spectrum of periodic *real* topological K-theory. A lot of effort was devoted to describe $THH(ku)$, where ku is connective complex K-theory [**Aus05**]: that case is markedly harder. It should also be noted that, rationally, the algebraic K-theory $K(KU)$ was determined in [**AR12**, Theorem 3.6].

All in all, to the best of our knowledge, a description of the spectrum $THH(KU)$ has not been given. Moreover, all the previous cited results do not deal with the multiplicative structure on THH. We do so here: we describe $THH(KU)$ as a commutative KU-algebra, by using different methods to the ones used by the previous authors. We give two descriptions: the first one is obtained in **Theorem 6.22**:

$$THH(KU) \simeq KU[K(\mathbb{Z},3)],$$

where the underlying KU-module of $KU[K(\mathbb{Z},3)]$ is $KU \wedge K(\mathbb{Z},3)_+$. The second one is given in **Theorem 6.27**: there is a morphism of commutative KU-algebras

$$\tilde{f} : F(\Sigma KU_{\mathbb{Q}}) \to THH(KU)$$

which is a weak equivalence. Here $F(\Sigma KU_{\mathbb{Q}})$ is the free commutative KU-algebra on the KU-module $\Sigma KU_{\mathbb{Q}}$. Moreover, $F(\Sigma KU_{\mathbb{Q}})$ is weakly equivalent as a commutative KU-algebra to the split square-zero extension $KU \vee \Sigma KU_{\mathbb{Q}}$.

We then determine $THH^n(KU)$. The first expression we gave above for $THH(KU)$ directly generalizes: one replaces $K(\mathbb{Z},3)$ by a suitable product of integral Eilenberg-Mac Lane spaces. See **Corollary 6.45**. This determines $THH^n(KU)$ as a commutative KU-algebra.

The expression $KU \vee \Sigma KU_{\mathbb{Q}}$ for $THH(KU)$ also generalizes to $THH^n(KU)$. In this case, the augmentation ideal $\overline{THH}^n(KU)$ is still rational, but it has a non-trivial non-unital commutative KU-algebra structure. We describe the non-unital commutative $\mathbb{Q}[t^{\pm 1}]$-algebra $\overline{THH}^n_*(KU)$ as iterated Hochschild homology. See **Theorem 6.56**.

Finally, we shift our attention to $X \otimes KU$, where X is a pointed CW-complex which is a reduced suspension, e.g. a sphere. We extend the stable equivalence of Theorem 6.27 to a morphism of commutative KU-algebras

$$F(X \wedge KU_{\mathbb{Q}}) \to X \otimes KU$$

which is a weak equivalence. This is **Theorem 6.73**.

Throughout, we use the model for KU given by Snaith [**Sna79**], [**Sna81**], namely $\Sigma^\infty_+ K(\mathbb{Z},2)[x^{-1}]$. We thus need to pay close attention to the process of inverting a homotopy element in a spectrum. We review this theory, develop the necessary results, and we prove in **Corollary 6.15** that THH commutes, as a commutative algebra, with localization at an element.

An adaptation of most of the contents of Chapters 2 to 5 appears in [**Sto18**]. The results of Chapter 6 are currently being recast into an article in preparation [**Sto**].

CHAPTER 1

Symmetric monoidal categories

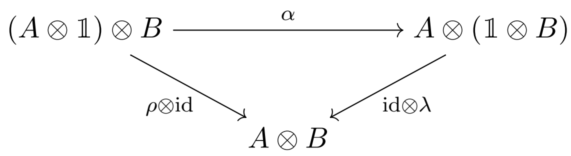
In this chapter we introduce symmetric monoidal categories and the part of their theory which will be useful to our purposes. Most of the material here is well-known: our main references were [**AM10**], [**Bor94**] and [**Lei04**]. We have included several proofs that are often left to the reader in the literature. There are also some (rather unsurprising) results such as Proposition 1.52 which we failed to find stated elsewhere. The examples most relevant to us can be found in Chapter 5.

1. Main definitions and coherence

If we are sloppy about size issues, we can say that there is a 2-category of categories, functors and natural transformations. We now introduce some 2-categories with objects the symmetric monoidal categories. It is useful but not indispensable to use the language of 2-categories: since we are using it merely as theoretical guidance, we do not include a discussion of it. The reader can consult [**AM10**, Appendix C] for a quick introduction to them.

Definition 1.1. A *monoidal category* is a category $\mathcal{V}$ with the extra structure given by a *monoidal product* (or *tensor product*) bifunctor $-\otimes- : \mathcal{V}\times\mathcal{V}\to\mathcal{V}$, a *unit* object $\mathbb{1}\in\mathcal{V}$, an *associator* natural isomorphism $\alpha : (-\otimes-)\otimes- \Rightarrow -\otimes(-\otimes-)$, and *unitor* natural isomorphisms $\lambda : \mathbb{1}\otimes- \Rightarrow -$ and $\rho : -\otimes\mathbb{1}\Rightarrow -$.

We ask that, for all $A, B, C, D \in \mathcal{V}$, the following diagrams of unitality and associativity commute.



$$
\begin{array}{ccccc}
 & & (A \otimes B) \otimes (C \otimes D) & & \\
 & \overset{\alpha}{\nearrow} & & \overset{\alpha}{\searrow} & \\
((A \otimes B) \otimes C)) \otimes D & & & & (A \otimes (B \otimes (C \otimes D))) \\
\Big\downarrow{\scriptstyle \alpha\otimes\mathrm{id}} & & & & \Big\uparrow{\scriptstyle \mathrm{id}\otimes\alpha} \\
(A \otimes (B \otimes C)) \otimes D & & \xrightarrow[\alpha]{\qquad\qquad\qquad} & & A \otimes ((B \otimes C) \otimes D)
\end{array}
$$

We will exclusively deal with *symmetric monoidal categories*: these are monoidal categories $\mathcal{V}$ together with a *symmetry* natural isomorphism $\sigma : - \otimes - \Rightarrow (- \otimes -) \circ \mathrm{twist}$, where $\mathrm{twist} : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \times \mathcal{V}$ switches the order of the factors. We ask that for all $A, B, C \in \mathcal{V}$ the following diagrams commute.

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{\sigma} & B \otimes A \\
 & {\scriptstyle \mathrm{id}}\searrow & \Big\downarrow{\scriptstyle \sigma} \\
 & & A \otimes B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \otimes \mathbb{1} & \xrightarrow{\sigma} & \mathbb{1} \otimes A \\
 & {\scriptstyle \rho}\searrow & \Big\downarrow{\scriptstyle \lambda} \\
 & & A
\end{array}
$$

$$
\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha} & A \otimes (B \otimes C) & \xrightarrow{\sigma} & (B \otimes C) \otimes A \\
\Big\downarrow{\scriptstyle \sigma\otimes\mathrm{id}} & & & & \Big\downarrow{\scriptstyle \alpha} \\
(B \otimes A) \otimes C & \xrightarrow[\alpha]{} & B \otimes (A \otimes C) & \xrightarrow[\mathrm{id}\otimes\sigma]{} & B \otimes (C \otimes A)
\end{array}
$$

REMARK 1.2. Kelly [**Kel64**] proved that $\rho : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ equals $\lambda : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$. In particular, since $\lambda \circ \sigma = \rho$ and λ is invertible, we have that $\sigma_{\mathbb{1},\mathbb{1}} = \mathrm{id}_{\mathbb{1}\otimes\mathbb{1}}$.

The commutative diagrams appearing in the previous definition, called *coherence conditions*, are there because we are not insisting that α, λ, ρ and σ be identities. If that were the case, we would say that $\mathcal{V}$ is *strict*, and then the diagrams would automatically commute. Note that a strict (symmetric) monoidal category is formally similar to a (commutative) monoid, that is, a set with a binary multiplication and a unit element satisfying associativity and unitality axioms (and commutativity). This explains the name.

However, strict monoidal categories rarely appear in nature. Instead of asking for identities, we are asking for natural isomorphisms: we are using the 2-categorical structure present in the realm of categories. But then we need to take care of coherence.

Note that the above coherence conditions form a fairly small list. However, they are sufficient to prove that any *formal* diagram that one might form starting from a finite list of objects of $\mathcal{V}$ and inserting $\mathbb{1}$'s and parenthesis, and going around by tensoring $\alpha, \lambda, \rho, \sigma$, their inverses, and identities, is commutative. This is one possible statement for Mac Lane's coherence theorem [**ML98**, VII.2]. The formality of a diagram, which we will not define, is necessary: consider for example the diagram $A \otimes A \overset{\sigma}{\underset{\mathrm{id}}{\rightrightarrows}} A \otimes A$. Note that Mac Lane's original list of axioms was longer: the current form is due to Kelly [**Kel64**]. One conclusion is that we can unambiguously suppress the parentheses in an expression involving a tensor of three or more objects.

In other words, any way of defining a functor $\otimes^n : \mathcal{V}^n \to \mathcal{V}$ by using the above procedure will be naturally isomorphic to any other such choice, and any diagram that one can cook up between these two choices by the same means is going to commute. All in all, starting from all the data above, one can make choices of functors $\otimes^n : \mathcal{V}^n \to \mathcal{V}$.

One can adopt the point of view that the definition just given is biased. Why single out a *binary* monoidal product and a *unit* object (to be interpreted as a 0-ary operation), then use these to artificially choose $\otimes^n$ functors and prove that the choice is immaterial? Leinster [**Lei04**, 3.1] gives a definition of an *unbiased monoidal category*. It is a category $\mathcal{V}$ together with a functor $\otimes^n : \mathcal{V}^n \to \mathcal{V}$ for every $n \geq 0$, and higher associators and unitors satisfying making all the possible diagrams of associativity and unitality commute.

Leinster proves that both definitions are equivalent. The first definition has the advantage of being much easier to verify in practice. One could further ask the question: what is special about the values $n = 0$ and $n = 2$ that restricting $\otimes^n$ to these values is sufficient to recover the whole "unbiased" structure? Leinster also answers this question [**Lei04**, 3.2].

We have defined symmetric monoidal categories: these are the 0-cells of the 2-category we are introducing. It turns out that there are different classes of 1-cells we will be dealing with. We will introduce the *lax symmetric*, *colax symmetric* and *strong symmetric* varieties: the first two can be *normal*.

Definition 1.3. Let $\mathcal{V}$ and $\mathcal{W}$ be symmetric monoidal categories. A *lax monoidal functor* from $\mathcal{V}$ to $\mathcal{W}$ is the data of a functor $F : \mathcal{V} \to \mathcal{W}$, *multiplication* (or *structure*) morphisms

$$\nabla_{A,B} : FA \otimes FB \to F(A \otimes B)$$

natural in $A, B \in \mathcal{V}$, and a unit map

$$\nabla_0 : \mathbb{1}_{\mathcal{W}} \to F(\mathbb{1}_{\mathcal{V}}).$$

We ask that, for all $A, B, C \in \mathcal{V}$, the following diagrams of associativity and unitality commute.

$$\begin{array}{ccccc} (FA \otimes FB) \otimes FC & \xrightarrow{\nabla \otimes \mathrm{id}} & F(A \otimes B) \otimes FC & \xrightarrow{\nabla} & F((A \otimes B) \otimes C) \\ \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle F\alpha} \\ FA \otimes (FB \otimes FC) & \xrightarrow[\mathrm{id} \otimes \nabla]{} & FA \otimes F(B \otimes C) & \xrightarrow[\nabla]{} & F(A \otimes (B \otimes C)) \end{array}$$

$$\begin{array}{ccccc} FA\otimes \mathbb{1} & \xrightarrow{\mathrm{id}\otimes\nabla_0} & FA\otimes F\mathbb{1} & \xrightarrow{\nabla} & F(A\otimes\mathbb{1}) \\ & \searrow^{\rho} & & \swarrow_{F\rho} & \\ & & FA & & \end{array}$$

$$\begin{array}{ccccc} \mathbb{1}\otimes FA & \xrightarrow{\nabla_0\otimes\mathrm{id}} & F\mathbb{1}\otimes FA & \xrightarrow{\nabla} & F(\mathbb{1}\otimes A) \\ & \searrow^{\lambda} & & \swarrow_{F\lambda} & \\ & & FA & & \end{array}$$

We say that F is *symmetric* if the following diagram of symmetry commutes. Our monoidal functors will generally be symmetric.

$$\begin{array}{ccc} FA\otimes FB & \xrightarrow{\sigma} & FB\otimes FA \\ \nabla\downarrow & & \downarrow\nabla \\ F(A\otimes B) & \xrightarrow[F\sigma]{} & F(B\otimes A) \end{array}$$

A lax (symmetric) monoidal functor is *normal* if ∇_0 is an isomorphism, and it is *strong* if ∇_0 and $\nabla_{A,B}$ are isomorphisms, for all $A, B \in \mathcal{V}$.

We will also need the notion of a *colax (symmetric) monoidal functor*: in this case, the source and target of $\nabla_{A,B}$ and of ∇_0 are reversed. It is *normal* if ∇_0 is an isomorphism.

Note that if a functor $F : \mathcal{V} \to \mathcal{W}$ has a strong symmetric monoidal structure, then it can also be given a colax symmetric monoidal structure, with structure morphisms given by the inverses of ∇ and ∇_0.

These diagrams are coherence conditions, and just as before, they are there because we are not demanding that $\alpha, \lambda, \rho, \sigma, \nabla$ and ∇_0 be identities, in which case F would be a *strict symmetric monoidal functor* between strict symmetric monoidal categories. Such a functor is formally similar to a morphism of commutative monoids. But as strict symmetric monoidal categories, they rarely appear in nature.

Mac Lane proved a coherence theorem for lax monoidal functors [**ML98**, XI.2], which implies that all the different ways of going from $FA_1 \otimes \cdots \otimes FA_n$ to $F(A_1 \otimes \cdots \otimes A_n)$ by means of inserting parenthesis, tensoring $\mathbb{1}$'s, $\alpha, \lambda, \rho, \sigma, \nabla$ and ∇_0, their inverses (when available) and identities, coincide. We denote it again by ∇ or by ∇_n if we want to be precise.

Just as there are unbiased symmetric monoidal categories, there are unbiased versions of the monoidal functors defined above; see [**Lei04**, 3.1.3].

Finally, we must define the 2-cells of the 2-categories of symmetric monoidal functors, for all of the variants defined above.

DEFINITION 1.4. A *monoidal transformation* of lax symmetric monoidal functors $F, G : \mathcal{V} \to \mathcal{W}$ is a natural transformation

$$\mathcal{V} \underset{G}{\overset{F}{\genfrac{}{}{0pt}{}{\longrightarrow}{\longrightarrow}}} \mathcal{W} \quad (\Downarrow \tau)$$

such that the following diagrams commute for every $A, B \in \mathcal{V}$.

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{\nabla^F} & F(A \otimes B) \\ {\scriptstyle \tau_A \otimes \tau_B}\downarrow & & \downarrow{\scriptstyle \tau_{A \otimes B}} \\ GA \otimes GB & \xrightarrow[\nabla^G]{} & G(A \otimes B). \end{array} \qquad \begin{array}{ccc} & & F\mathbb{1} \\ & \nearrow^{\nabla_0^F} & \\ \mathbb{1} & & \downarrow{\scriptstyle \tau_{\mathbb{1}}} \\ & \searrow_{\nabla_0^G} & \\ & & G\mathbb{1} \end{array}$$

The definition of a monoidal transformation between strong or normal lax symmetric monoidal functors is the same.

One can also define a monoidal transformation between colax symmetric monoidal functors: in this case, the direction of the ∇ and ∇_0 in the diagrams is reversed.

A *monoidal isomorphism* is a monoidal transformation which is a natural isomorphism.

Note that there is no need to add the adjective "symmetric" to monoidal transformations: there is no condition on τ involving the symmetries.

There are several choices of 1-cells for 2-categories having as 0-cells the symmetric monoidal categories and as 2-cells the monoidal transformations. For example, we can consider lax symmetric monoidal functors, colax symmetric monoidal functors and strong symmetric monoidal functors. The notations $\mathbf{SMCat_{lax}}, \mathbf{SMCat_{colax}}, \mathbf{SMCat_{str}}$ denote the respective 2-categories.

Since the 2-categorical language is for us a mere guide, we will not do all the verifications required. They are: checking that the vertical composition of monoidal transformations is monoidal, associative and unital; checking that the composition of two 1-cells is a 1-cell; checking that the horizontal composition of monoidal transformations is monoidal, associative and unital; and checking that horizontal and vertical composition satisfy the interchange law. Note that if $\mathcal{V} \xrightarrow{F} \mathcal{W} \xrightarrow{G} \mathcal{U}$ are lax symmetric monoidal functors, then $G \circ F$ is symmetric monoidal, with structure morphisms given by

$$GFA \otimes GFB \xrightarrow{\nabla^G} G(FA \otimes FB) \xrightarrow{G\nabla^F} GF(A \otimes B) \qquad \text{and}$$

$$\mathbb{1} \xrightarrow{\nabla_0^G} G\mathbb{1} \xrightarrow{G\nabla_0^F} GF\mathbb{1}.$$

The last of our remarks on coherence is the following lemma, which will be used later. The notation $\mathcal{U}^n$ where $\mathcal{U}$ is a symmetric monoidal category denotes the n-fold cartesian product of categories $\mathcal{U} \times \cdots \times \mathcal{U}$.

LEMMA 1.5. *Let $F : \mathcal{V} \to \mathcal{W}$ be a lax symmetric monoidal functor between symmetric monoidal categories. There are lax symmetric monoidal functors and a monoidal transformation*

$$\begin{array}{ccccc} & & \mathcal{W}^n & & \\ & \overset{F^n}{\nearrow} & & \overset{\otimes_n}{\searrow} & \\ \mathcal{V}^n & & \Downarrow \nabla_n & & \mathcal{W} \\ & \underset{\otimes_n}{\searrow} & & \underset{F}{\nearrow} & \\ & & \mathcal{V} & & \end{array}$$

for every $n \geq 1$. The functors are strong if F is strong.

In particular, there are monoidal transformations

$$\mathcal{V} \underset{F((-)^{\otimes n})}{\overset{F(-)^{\otimes n}}{\Downarrow \nabla_n}} \mathcal{W}.$$

between lax symmetric monoidal functors, which are strong if F is strong.

PROOF. It is straightforward to give a symmetric monoidal structure to $\mathcal{V}^n$ and $\mathcal{W}^n$, and a lax symmetric monoidal structure to the functor $F^n : \mathcal{V}^n \to \mathcal{W}^n$.

To prove the functor $\otimes_n$ is strong symmetric monoidal, it suffices to show this is the case for $\otimes_2 = \otimes$, since the $\otimes_n$ for higher n are built from the $n = 2$ one by tensoring it with identities and associators, which are strong symmetric monoidal. This is a straightforward verification, carried out in [**JS93**, Proposition 5.4].

To prove that ∇_n is monoidal, it suffices to show this is the case for $n = 2$, since the natural transformation for higher n is built by composing and tensoring the $n = 2$ one with identities and associators, which are symmetric monoidal. This is a straightforward, if lengthy, diagram chase, obtained by using the properties of naturality, associativity, unitality and symmetry of ∇.

The particular case is obtained by prewhiskering with the iterated diagonal functor $\mathcal{V} \to \mathcal{V}^n$, which is strong symmetric monoidal. Indeed, whiskering preserves monoidality of natural transformations [**AM10**, 3.21, 3.24]. □

REMARK 1.6. This remark is essentially due to Mike Shulman [**Shu**]; we include it because it is interesting, but it will not be used in the rest of the dissertation. Both the statement that we can define $\otimes_n$ and that it is strong symmetric monoidal, and the statement that we can define ∇_n and that it is symmetric monoidal are instances of the same result. Lack [**Lac00**, 3.6] proved a coherence theorem for pseudomonoids X in monoidal bicategories $\mathcal{C}$. By means of it, one obtains a 1-cell $\mu_n : X^n \to X$. Moreover, when $\mathcal{C}$ and X are symmetric, then the multiplication $\mu : X \otimes X \to X$ is strong symmetric monoidal (i.e. a strong symmetric morphism of pseudomonoids), therefore μ_n is strong symmetric, too. This sole result encompasses both of the above, by taking $\mathcal{C}$ to be the 2-category with products **Cat** for the first case, and by taking $\mathcal{C}$ to be $\mathbf{Oplax}(\mathbf{2}, \mathbf{Cat})$, the 2-category with products consisting of oplax functors from the interval category to **Cat** for the second case.

An equivalence of symmetric monoidal categories is an invertible 2-cell in $\mathbf{SMCat_{str}}$:

DEFINITION 1.7. Let $F : \mathcal{V} \to \mathcal{W}$ be a strong symmetric monoidal functor. We say it is an *equivalence of symmetric monoidal categories* if there exists a strong symmetric monoidal functor $G : \mathcal{W} \to \mathcal{V}$ and monoidal isomorphisms $\mathrm{id} \Rightarrow GF$ and $FG \Rightarrow \mathrm{id}$. In this case, we say that $\mathcal{V}$ and $\mathcal{W}$ are *monoidally equivalent.*

PROPOSITION 1.8. *A strong symmetric monoidal functor $F : \mathcal{V} \to \mathcal{W}$ is an equivalence of symmetric monoidal categories if and only if it is an equivalence of ordinary categories.*

PROOF. If F is an equivalence of categories, then it is part of an adjoint equivalence $(F, G, \eta, \varepsilon)$. The functor G gets a strong symmetric monoidal structure: $\nabla^G : GA \otimes GB \to G(A \otimes B)$ is defined as the adjoint to the isomorphism

$$F(GA \otimes GB) \xrightarrow{(\nabla^F)^{-1}} FGA \otimes FGB \xrightarrow{\varepsilon \otimes \varepsilon} A \otimes B\ ,$$

and the unit of G is the adjoint to the inverse of ∇^F_0. One then verifies that η and ε are monoidal transformations. □

We finish with the following

PROPOSITION 1.9. *Let $\mathcal{V}$ and $\mathcal{W}$ be symmetric monoidal categories. The categories* $\mathbf{Lax}(\mathcal{V}, \mathcal{W})$ *(resp.* $\mathbf{SLax}(\mathcal{V}, \mathcal{W})$*) of lax monoidal functors (resp. lax symmetric monoidal functors) from $\mathcal{V}$ to $\mathcal{W}$ have symmetric monoidal structures.*

PROOF. Let $F, G : \mathcal{V} \to \mathcal{W}$ be lax monoidal functors. We define $F \otimes G : \mathcal{V} \to \mathcal{W}$ by $(F \otimes G)(A) = FA \otimes GA$. We define $\mathbb{1} \in \mathbf{Lax}(\mathcal{V}, \mathcal{W})$ to be the constant functor at $\mathbb{1}_{\mathcal{W}}$. The functor $F \otimes G$ is lax monoidal, with structure morphisms given by

$$FA \otimes GA \otimes FB \otimes GB \xrightarrow{\mathrm{id} \otimes \sigma \otimes \mathrm{id}} FA \otimes FB \otimes GA \otimes GB \xrightarrow{\nabla^F \otimes \nabla^G} F(A \otimes B) \otimes G(A \otimes B)$$

and unit $\mathbb{1} \to F \otimes G$ given by $\mathbb{1} \xrightarrow{\lambda^{-1}} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\nabla_0 \otimes \nabla_0} FA \otimes GA$.

If F and G are symmetric, then so is $F \otimes G$. □

2. Monoids

Let $\mathcal{V}$ be a symmetric monoidal category. We can internalize the notion of a monoid inside of $\mathcal{V}$.

DEFINITION 1.10. A *monoid* in $\mathcal{V}$ is an object $M \in \mathcal{V}$ together with a *multiplication* morphism $\mu : M \otimes M \to M$ and a *unit* morphism $\eta : \mathbb{1} \to M$ such that the following diagrams of associativity and unitality commute.

$$\begin{array}{ccccc} (M\otimes M)\otimes M & \xrightarrow{\alpha} & M\otimes(M\otimes M) & \xrightarrow{\mathrm{id}\otimes\mu} & M\otimes M \\ {\scriptstyle \mu\otimes\mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle\mu} \\ M\otimes M & & \xrightarrow[\mu]{\qquad\qquad} & & M \end{array}$$

$$\begin{array}{ccccc} \mathbb{1}\otimes M & \xrightarrow{\eta\otimes\mathrm{id}} & M\otimes M & \xleftarrow{\mathrm{id}\otimes\eta} & M\otimes\mathbb{1} \\ & {\scriptstyle\lambda}\searrow & {\scriptstyle\mu}\downarrow & \swarrow{\scriptstyle\rho} & \\ & & M & & \end{array}$$

It is *commutative* if the following diagram commutes.

$$\begin{array}{ccc} M\otimes M & \xrightarrow{\sigma} & M\otimes M \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & M & \end{array}$$

Let M and N be monoids, possibly commutative. A morphism $f : M \to N$ is a *morphism of monoids* if the following diagrams commute.

$$\begin{array}{ccc} M\otimes M & \xrightarrow{f\otimes f} & N\otimes N \\ {\scriptstyle\mu_M}\downarrow & & \downarrow{\scriptstyle\mu_N} \\ M & \xrightarrow[f]{} & N \end{array} \qquad\qquad \begin{array}{ccc} & & M \\ & {\scriptstyle\eta_M}\nearrow & \\ \mathbb{1} & & \downarrow{\scriptstyle f} \\ & {\scriptstyle\eta_N}\searrow & \\ & & N \end{array}$$

Thus we obtain categories $\mathbf{Mon}(\mathcal{V})$ and $\mathbf{CMon}(\mathcal{V})$ of monoids and commutative monoids in $\mathcal{V}$, respectively.

PROPOSITION 1.11. *The symmetric monoidal structure of $\mathcal{V}$ induces symmetric monoidal structures on* $\mathbf{Mon}(\mathcal{V})$ *and* $\mathbf{CMon}(\mathcal{V})$.

PROOF. We give the tensor product of two (commutative) monoids the (commutative) monoid structure with multiplication

$$M\otimes N\otimes M\otimes N \xrightarrow{\mathrm{id}\otimes\sigma\otimes\mathrm{id}} M\otimes M\otimes N\otimes N \xrightarrow{\mu\otimes\mu} M\otimes N$$

and unit

$$\mathbb{1} \xrightarrow{\rho^{-1}} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\eta \otimes \eta} M \otimes N.$$

The unit of $\mathbf{Mon}(\mathcal{V})$ and $\mathbf{CMon}(\mathcal{V})$ is to be $\mathbb{1}$, which is a commutative monoid with multiplication $\lambda : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ and unit $\mathrm{id} : \mathbb{1} \to \mathbb{1}$. The associator, unitor and symmetry morphisms are to be the ones of $\mathcal{V}$, which one proves to be morphisms of monoids.

As a sample for the things that would need verification, let us prove that if M and N are commutative, then so is $M \otimes N$; in a similar fashion, one proves associativity and unitality. The proof is the commutativity of the following diagram. The upper square commutes by coherence, and the lower square commutes by commutativity of M and of N.

$$\begin{array}{ccc}
M \otimes N \otimes M \otimes N & \xrightarrow{\quad\sigma\quad} & M \otimes N \otimes M \otimes N \\
\downarrow{\scriptstyle \mathrm{id}\otimes\sigma\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\sigma\otimes\mathrm{id}} \\
M \otimes M \otimes N \otimes N & \xrightarrow{\quad\sigma\otimes\sigma\quad} & M \otimes M \otimes N \otimes N \\
& {\scriptstyle \mu\otimes\mu} \searrow \quad \swarrow {\scriptstyle \mu\otimes\mu} & \\
& M \otimes N &
\end{array}$$

□

Remark 1.12. Let M be a monoid in $\mathcal{V}$. There is an *opposite monoid* M^{op} with the same unit and with multiplication given by

$$M \otimes M \xrightarrow{\sigma} M \otimes M \xrightarrow{\mu} M.$$

Then M is commutative if and only if this multiplication equals μ, i.e. if and only if $M = M^{\mathrm{op}}$ as monoids.

We have just seen that the tensor product of two monoids is a monoid: define the *enveloping monoid* of M to be $M^e = M \otimes M^{\mathrm{op}}$.

This is the first step in the definition of a 2-functor

$$\mathbf{CMon} : \mathbf{SMCat_{lax}} \to \mathbf{SMCat_{lax}}$$

(and similarly for non-commutative monoids, but for conciseness we will stick to the commutative case). We will also have a 2-functor $\mathbf{CMon} : \mathbf{SMCat_{str}} \to \mathbf{SMCat_{str}}$.

We have seen how a 0-cell gets sent to a 0-cell. We will now analyze the behavior for 1-cells and 2-cells.

Proposition 1.13. *Let $F : \mathcal{V} \to \mathcal{W}$ be a lax symmetric monoidal functor. There is an induced lax symmetric monoidal functor*

$$F : \mathbf{CMon}(\mathcal{V}) \to \mathbf{CMon}(\mathcal{W})$$

which is strong if F is strong.

Let $\mathcal{V} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow\tau}}}} \mathcal{W}$ *be a monoidal transformation between lax symmetric monoidal functors. There is an induced monoidal transformation between lax symmetric monoidal functors*

$$\mathbf{CMon}(\mathcal{V}) \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow\tau}}}} \mathbf{CMon}(\mathcal{W})\ .$$

Proof. If $M \in \mathbf{CMon}(\mathcal{V})$, then $FM \in \mathbf{CMon}(\mathcal{W})$ with multiplication

$$FM \otimes FM \xrightarrow{\nabla} F(M \otimes M) \xrightarrow{F\mu} FM \qquad \text{and unit}$$

$$\mathbb{1} \xrightarrow{\nabla_0} F\mathbb{1} \xrightarrow{F\eta} FM.$$

The lax symmetric monoidal structure of F is given by ∇ and ∇_0; we need to check that they are morphisms of monoids. This is the commutativity of the following diagram, plus a similar check for the unit condition.

$$\begin{array}{ccc}
FM \otimes FM \otimes FM \otimes FM & \xrightarrow{\nabla\otimes\nabla} & F(M \otimes M) \otimes F(M \otimes M) \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\sigma\otimes id} & & \Big\downarrow{\scriptstyle \nabla} \\
FM \otimes FM \otimes FM \otimes FM & & F(M \otimes M \otimes M \otimes M) \\
\Big\downarrow{\scriptstyle \nabla\otimes\nabla} & & \Big\downarrow{\scriptstyle F(\mathrm{id}\otimes\sigma\otimes\mathrm{id})} \\
F(M \otimes M) \otimes F(M \otimes M) & & F(M \otimes M \otimes M \otimes M) \\
\Big\downarrow{\scriptstyle F\mu\otimes F\mu} & & \Big\downarrow{\scriptstyle F(\mu\otimes\mu)} \\
FM \otimes FM & \xrightarrow[\nabla]{} & F(M \otimes M)
\end{array}$$

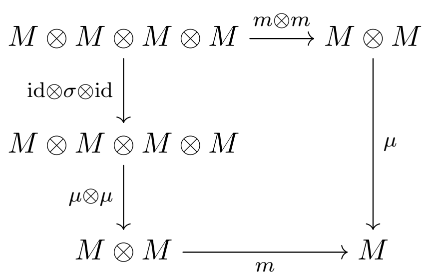
The 2-cell assertion is automatic. $\square$

We will frequently use the following theorem. Note that we really mean *isomorphisms* of categories and not merely equivalences.

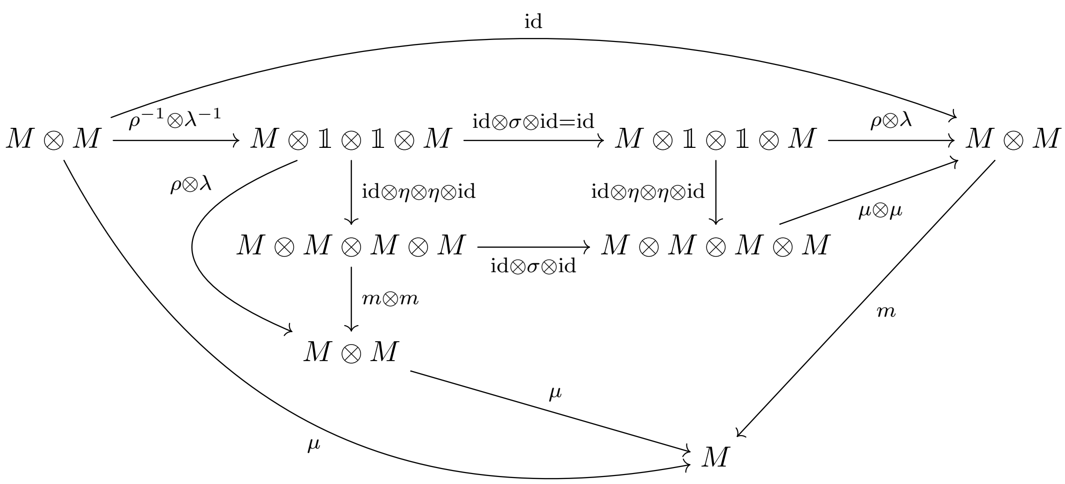
Theorem 1.14 (Eckmann-Hilton argument). *The two units and multiplications of a monoid in* $\mathbf{Mon}(\mathcal{V})$ *coincide and are commutative. Therefore, the symmetric monoidal categories* $\mathbf{Mon}(\mathbf{Mon}(\mathcal{V}))$ *and* $\mathbf{CMon}(\mathcal{V})$ *are isomorphic via a strict symmetric monoidal functor.*

Similarly, $\mathbf{CMon}(\mathbf{Mon}(\mathcal{V}))$ *and* $\mathbf{CMon}(\mathbf{CMon}(\mathcal{V}))$ *are isomorphic to* $\mathbf{CMon}(\mathcal{V})$ *via strict symmetric monoidal functors.*

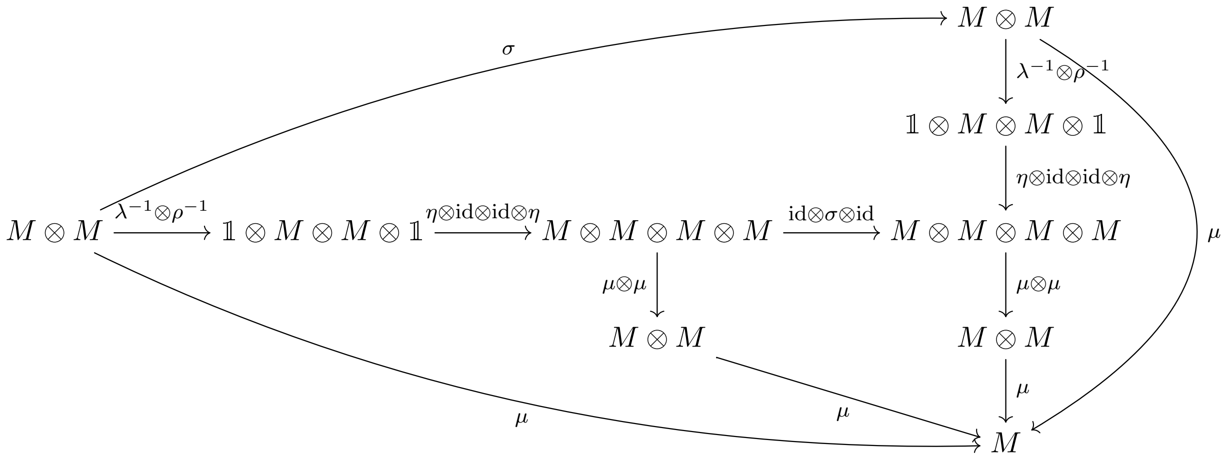
PROOF. Let M be an object of $\mathbf{Mon}(\mathbf{Mon}(\mathcal{V}))$. This means that $M_1 = (M, \mu, \eta)$ is a monoid, $M_2 = (M, m, e)$ is a monoid, and $m : M_1 \otimes M_1 \to M_1$, $e : \mathbb{1} \to M_1$ are morphisms of monoids. The fact that e preserves the unit of M_1 directly implies that $e = \eta$. Let us now check that $\mu = m$. First note that, as m is a morphism for M_1, the following diagram commutes. It is called the *exchange law.*



The equation $\mu = m$ can be expressed as the commutativity of the outer part of the following diagram. It is commutative because all the inner parts commute: this follows from η being the unit both for M_1 and for M_2, the exchange law, σ being natural and $\sigma_{\mathbb{1},\mathbb{1}} = \mathrm{id}_{\mathbb{1}\otimes\mathbb{1}}$ (Lemma 1.2).



The commutativity of μ is the commutativity of the outer part of the following diagram. It follows from η being the unit of μ, the exchange law and coherence.



Conversely, if (M, μ, η) is a commutative monoid, then similarly one checks that μ and η are morphisms of monoids. Therefore, the obvious functors between $\mathbf{Mon}(\mathbf{Mon}(\mathcal{V}))$ and $\mathbf{CMon}(\mathcal{V})$ are inverses to each other and are strict symmetric monoidal.

For $\mathbf{Mon}(\mathbf{CMon}(\mathcal{V}))$ and $\mathbf{CMon}(\mathbf{CMon}(\mathcal{V}))$ the proof is the same. □

We finish with a reformulation of the notion of monoid from [**AM10**, 3.4], due to Bénabou, which is conceptually appealing and will be useful later. Denote by $\mathbf{1}$ the category with one object $*$ and one identity arrow: it has a canonical symmetric monoidal structure.

Proposition 1.15. *There are strict symmetric monoidal isomorphisms of categories between* $\mathbf{Mon}(\mathcal{V})$ *and* $\mathbf{Lax}(\mathbf{1}, \mathcal{V})$*, and between* $\mathbf{CMon}(\mathcal{V})$ *and* $\mathbf{SLax}(\mathbf{1}, \mathcal{V})$*.*

Proof. If $M \in \mathbf{Mon}(\mathcal{V})$, define $F_M : \mathbf{1} \to \mathcal{V}$ as $* \mapsto M$. It has a lax monoidal structure given by the monoid structure of M. The monoid M is commutative if and only if F_M is symmetric.

Conversely, if $F : \mathbf{1} \to \mathcal{V}$ is lax monoidal, then the lax monoidal structure of F gives $M := F(*)$ the structure of a monoid, which is commutative if and only if F is symmetric. □

2.1. Comonoids. If $\mathcal{V}$ is a symmetric monoidal category, then the opposite category $\mathcal{V}^{\text{op}}$ has a natural symmetric monoidal structure. The definitions and results given here are dual to the ones just given, so we will be brief.

Definition 1.16. A (cocommutative) *comonoid* in $\mathcal{V}$ is an object $C \in \mathcal{V}$ together with a *comultiplication* morphism $\Delta : C \to C \otimes C$ and a *counit* morphism $\varepsilon : C \to \mathbb{1}$

such that the associativity, unitality (and commutativity) diagrams of Definition 1.10 commute, after replacing the labels μ by Δ, η by ε and α, ρ, λ by their inverses.

Let C and D be comonoids, possibly cocommutative. A morphism $f : C \to D$ is a *morphism of comonoids* if the diagrams of the definition of a morphism of monoids in Definition 1.10 commute, after replacing the multiplications and units by comultiplications and counits.

Thus, we obtain categories $\mathbf{Comon}(\mathcal{V})$ and $\mathbf{CoComon}(\mathcal{V})$ of comonoids and cocommutative comonoids in $\mathcal{V}$, respectively. They get induced symmetric monoidal structures. There is a 2-functor

$$\mathbf{CoComon} : \mathbf{SMCat}_{\mathbf{colax}} \to \mathbf{SMCat}_{\mathbf{colax}}$$

and also a 2-functor $\mathbf{CoComon} : \mathbf{SMCat}_{\mathbf{str}} \to \mathbf{SMCat}_{\mathbf{str}}$, since any strong monoidal functor gets a canonical colax structure. Thus, if $\mathcal{V} \underset{G}{\overset{F}{\underset{\Downarrow \tau}{\rightrightarrows}}} \mathcal{W}$ is a monoidal transformation between colax (or strong) symmetric monoidal functors, there is an induced monoidal transformation between colax (or strong) symmetric monoidal functors

$$\mathbf{CoComon}(\mathcal{V}) \underset{G}{\overset{F}{\underset{\Downarrow \tau}{\rightrightarrows}}} \mathbf{CoComon}(\mathcal{W}) \ .$$

3. Cartesian monoidal categories

Perhaps the simplest examples of symmetric monoidal categories come from categories with finite products. Since the details are usually skipped and since this example will be important for us, we feel it is a good opportunity to give a somewhat extended presentation.

Let $\mathcal{C}$ be a category with finite products. Choose a functor $- \times -$ right adjoint to $\operatorname{diag} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, choose a final object $\mathbb{1}$, and denote by $e : A \to \mathbb{1}$ the unique such morphism for each A. The unit of the adjunction gives maps $\Delta : A \to A \times A$ natural in $A \in \mathcal{C}$, called the *diagonal*. The counit of the adjunction gives maps $\varepsilon : (A \times B, A \times B) \to (A, B)$ natural in (A, B). We call the first component $\varepsilon_1 : A \times B \to A$ and the second component $\varepsilon_2 : A \times B \to B$: these are the *projections*.

The triangular identities for the adjunction give the following commutative diagrams for all A, B.

$$\begin{array}{ccc} A \xrightarrow{\Delta} A \times A & A \xrightarrow{\Delta} A \times A & A \times B \xrightarrow{\Delta} (A \times B) \times (A \times B) \\ {}_{\mathrm{id}} \searrow \quad \downarrow \varepsilon_1 & {}_{\mathrm{id}} \searrow \quad \downarrow \varepsilon_2 & {}_{\mathrm{id}} \searrow \quad \downarrow \varepsilon_1 \times \varepsilon_2 \\ A & A & A \times B \end{array} \tag{1.17}$$

Note as well that $\Delta : A \to A \times A$ is the unique such morphism making the left and center diagram commute, by the universal property of products.

We can explicitly define associators, unitors, and a symmetry by means of these maps. The associator is

$$(A \times B) \times C \xrightarrow{\Delta} ((A \times B) \times C) \times ((A \times B) \times C) \xrightarrow{\varepsilon_1\varepsilon_1 \times ((\varepsilon_2\varepsilon_1 \times \varepsilon_2) \circ \Delta)} A \times (B \times C).$$

The inverse to the right unitor is

$$A \xrightarrow{\Delta} A \times A \xrightarrow{\mathrm{id} \times e} A \times \mathbb{1}$$

and the left unitor is defined similarly. The symmetry is

$$A \times B \xrightarrow{\Delta} (A \times B) \times (A \times B) \xrightarrow{\varepsilon_2 \times \varepsilon_1} B \times A.$$

The proof of the following proposition is elementary, if long and tedious.

PROPOSITION 1.18. *Let $\mathcal{C}$ be a category with finite products. The structure maps defined above give $\mathcal{C}$ the structure of a symmetric monoidal category.*

DEFINITION 1.19. A symmetric monoidal category with structure maps given as above is called *cartesian*. A strong symmetric monoidal functor between cartesian categories is called *cartesian*.

Note that a cartesian functor preserves finite products.

The following proposition gives one possible characterization of cartesian categories among monoidal categories. Since we think it is interesting and we cannot find a published source for it (though the fact is known; the author learned of the statement through the *nLab*), we provide a proof. Let $\mathcal{V}$ be any symmetric monoidal category.

Denote by $D : \mathcal{V} \to \mathcal{V}$ the composition $\mathcal{V} \xrightarrow{\mathrm{diag}} \mathcal{V} \times \mathcal{V} \xrightarrow{-\otimes-} \mathcal{V}$. Note that it has a strong symmetric monoidal structure with unit $\nabla_0 = \lambda^{-1} = \rho^{-1} : \mathbb{1} \to \mathbb{1} \otimes \mathbb{1}$ and multiplication $\nabla_{A,B} :\ A \otimes A \otimes B \otimes B \xrightarrow{\mathrm{id} \otimes \sigma \otimes \mathrm{id}} A \otimes B \otimes A \otimes B$ for all $A, B \in \mathcal{V}$. Denote by $\mathbb{1} : \mathcal{V} \to \mathcal{V}$ the constant functor which maps any arrow in $\mathcal{V}$ to $\mathrm{id} : \mathbb{1} \to \mathbb{1}$, and give it the strong symmetric monoidal structure where $\nabla_0 = \mathrm{id}$ and $\nabla_{A,B} = \lambda = \rho$ for all $A, B \in \mathcal{V}$.

PROPOSITION 1.20. *Let $\mathcal{V}$ be a symmetric monoidal category. Then $\mathcal{V}$ is cartesian if and only if we have monoidal transformations*

$$\mathcal{V} \underset{D}{\overset{\mathrm{id}}{\Downarrow \Delta}} \mathcal{V} \qquad \textit{and} \qquad \mathcal{V} \underset{\mathbb{1}}{\overset{\mathrm{id}}{\Downarrow e}} \mathcal{V}$$

such that the following diagrams commute.

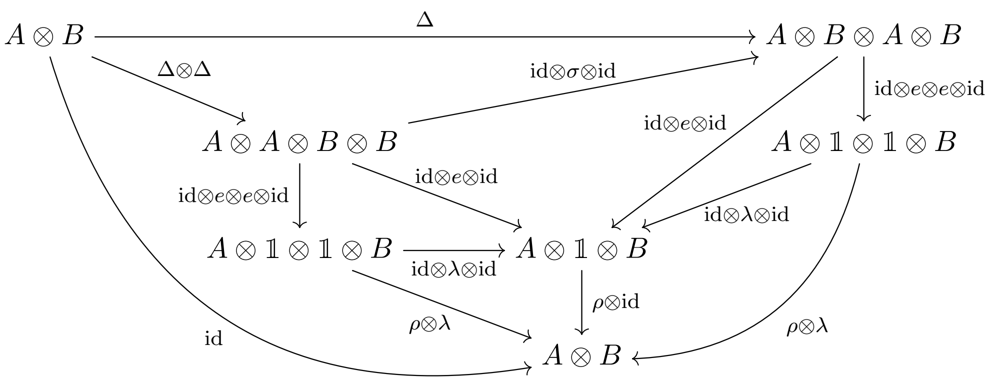
$$(1.21) \qquad \begin{array}{ccccc} A & \xrightarrow{\Delta} & A \otimes A & \xrightarrow{e \otimes \mathrm{id}} & \mathbb{1} \otimes A \\ & {\scriptstyle \mathrm{id}} \searrow & & & \downarrow \lambda \\ & & & & A \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{\Delta} & A \otimes A & \xrightarrow{\mathrm{id} \otimes e} & A \otimes \mathbb{1} \\ & {\scriptstyle \mathrm{id}} \searrow & & & \downarrow \rho \\ & & & & A \end{array}$$

PROOF. Suppose $\mathcal{V}$ is cartesian. Naturality and monoidality of e are evident. The commutativity of the diagrams (1.21) follows from the definition of λ and ρ. Monoidality of Δ is a diagram chase.

We now prove the converse. We first prove that the tensor product functor is right adjoint to diag : $\mathcal{V} \to \mathcal{V} \times \mathcal{V}$. To do this, we provide unit and counit natural transformations. The components of the unit are the $\Delta_A : A \to A \otimes A$. The components of the counit are the $\varepsilon_{A,B} : (A \otimes B, A \otimes B) \to (A, B)$ given in the first and second coordinates respectively by

$$A \otimes B \xrightarrow{\mathrm{id} \otimes e} A \otimes \mathbb{1} \xrightarrow{\rho} A \quad \text{and} \quad A \otimes B \xrightarrow{e \otimes \mathrm{id}} \mathbb{1} \otimes B \xrightarrow{\lambda} B.$$

One of the triangular identities follows directly from the commutativity of (1.21). The other triangular identity is the commutativity of the outer part of the following diagram.

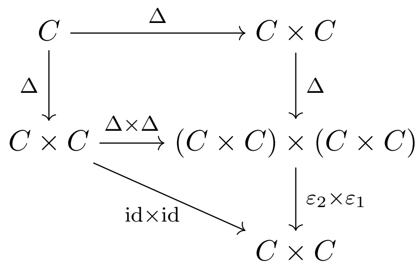


The commutativity of each part of the diagram follows from (1.21), monoidality of Δ and e, coherence and naturality of e. Note that the latter says exactly that for every $A \in \mathcal{V}$, there is a unique arrow $A \to \mathbb{1}$, namely e_A. In conclusion, $\mathcal{V}$ is cartesian. $\square$

Any object of a cartesian category has a unique cocommutative comonoid structure:

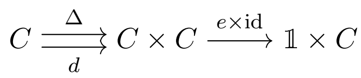
Proposition 1.22. *Let $\mathcal{C}$ be a cartesian category. Any $C \in \mathcal{C}$ has a unique comonoid structure, and it is cocommutative. The comultiplication is given by $\Delta : C \to C \times C$ and the counit by $e : C \to \mathbb{1}$. In particular, there is an isomorphism of categories $\mathcal{C} \cong \mathbf{CoComon}(\mathcal{C})$ which is strict symmetric monoidal.*

Proof. We first address the fact that (C, Δ, e) is a cocommutative comonoid. The counitality diagram commutes by commutativity of (1.21). The cocommutativity diagram is the outer part of the following diagram.

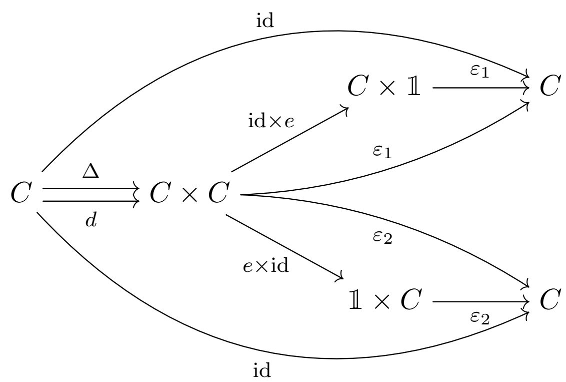


It commutes by naturality of Δ and (1.17). The coassociativity is another similar, if longer, diagram proof.

We now prove uniqueness. There is only one counit since $\mathbb{1}$ is final. Let $d : C \to C \times C$ be a comultiplication such that (C, d, e) is a comonoid. First observe that the counitality axiom, combined with the explicit definitions for ρ^{-1} and λ^{-1}, gives that the following diagram commutes, and similarly for $\mathrm{id} \times e$.



Together with the naturality of ε, this gives that the following diagram commutes. But it is a diagram for the universal property of the product, hence $\Delta = d$.



□

COROLLARY 1.23. *Let $F : \mathcal{C} \to \mathcal{V}$ be a colax symmetric monoidal functor from a cartesian category to a symmetric monoidal category. There is an induced colax symmetric monoidal functor*

$$\mathcal{C} \to \mathbf{CoComon}(\mathcal{V}) \tag{1.24}$$

which is cartesian if $\mathcal{C} \to \mathcal{V}$ is strong.

PROOF. Apply the dual of Proposition 1.13 to F and replace $\mathbf{CoComon}(\mathcal{C})$ by $\mathcal{C}$. □

Conversely to Proposition 1.22,

PROPOSITION 1.25. *Let $\mathcal{V}$ be a symmetric monoidal category. Then $\mathbf{CoComon}(\mathcal{V})$ is cartesian.*

PROOF. We use Proposition 1.20. As Δ we use the comultiplications, and as e we use the counits. This is legitimate, because the fact that the comonoids under consideration are cocommutative implies that Δ is a morphism of comonoids. Naturality of Δ and e follows from the definition of a morphism of comonoids. Monoidality of e is automatic, and monoidality of Δ is the definition of the comultiplication in a tensor product of comonoids. The diagrams (1.21) are the counitality conditions of a comonoid. □

REMARK 1.26. Note that, while $\mathbf{Comon}(\mathcal{V})$ is usually not cartesian, at least its unit $\mathbb{1}$ is a final object. Indeed, every comonoid C has a counit morphism $\varepsilon : C \to \mathbb{1}$, easily seen to be a morphism of comonoids, and if $f : C \to \mathbb{1}$ is another morphism of comonoids, then since it preserves the counit, it must be equal to ε.

REMARK 1.27. One can define a *cocartesian* category by replacing products with coproducts. Then one can prove duals of all the previous propositions. For example, a symmetric monoidal category of commutative monoids is cocartesian.

Moreover, similarly as in the previous remark, one proves that $\mathbf{Mon}(\mathcal{V})$, while not generally cocartesian, has $\mathbb{1}$ as an initial object.

In a cartesian category we can internalize groups: this is not possible in a mere symmetric monoidal category.

DEFINITION 1.28. Let $\mathcal{C}$ be a cartesian category. An object $A \in \mathcal{C}$ together with maps $\mu : A \times A \to A$, $\eta : \mathbb{1} \to A$ and $(-)^{-1} : A \to A$ is an *(abelian) group object* if (A, μ, η) is a (commutative) monoid in $\mathcal{C}$ and the following diagrams of left and right invertibility commute.

$$\begin{array}{ccccc} A & \xrightarrow{\Delta} & A \times A & \xrightarrow{(-)^{-1}\times \mathrm{id}} & A \times A \\ {\scriptstyle e}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ \mathbb{1} & & \xrightarrow{\quad\eta\quad} & & A \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{\Delta} & A \times A & \xrightarrow{\mathrm{id}\times(-)^{-1}} & A \times A \\ {\scriptstyle e}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ \mathbb{1} & & \xrightarrow{\quad\eta\quad} & & A \end{array} \tag{1.29}$$

If A and B are (abelian) group objects, an arrow $f : A \to B$ is a *morphism of groups* if it is a morphism of underlying monoids. We denote by $\mathbf{Ab}(\mathcal{C})$ the category of abelian group objects in $\mathcal{C}$.

More generally, in a cartesian category one can internalize more general algebraic theories, also called finite product theories or (possibly multi-sorted) Lawvere theories; see [**Bor94**, Chapter 3]. We shall not need this level of generality. However, we shall also make use of ($\mathbb{N}$-graded) ring objects, so now we introduce them.

Definition 1.30. Let $\mathcal{C}$ be a cartesian category. An object $R \in \mathcal{C}$ together with maps $\mu : R \times R \to R$, $\eta : \mathbb{1} \to R$, $(-)^{-1} : R \to R$, $m : R \times R \to R$ and $e : \mathbb{1} \to R$ is a *ring object* if $(R, \mu, \eta, (-)^{-1})$ is an abelian group object, (R, m, e) is a monoid, and the following distributivity diagrams commute.

$$\begin{array}{ccc}
R\times R\times R & \xrightarrow{\mathrm{id}\times\mu} & R\times R\\
\downarrow{\scriptstyle \Delta\times\mathrm{id}\times\mathrm{id}} & & \Big\downarrow\\
R\times R\times R\times R & & \\
\downarrow{\scriptstyle \mathrm{id}\times\sigma\times\mathrm{id}} & & {\scriptstyle m}\\
R\times R\times R\times R & & \\
\downarrow{\scriptstyle m\times m} & & \Big\downarrow\\
R\times R & \xrightarrow[\mu]{} & R
\end{array}
\qquad
\begin{array}{ccc}
R\times R\times R & \xrightarrow{\mu\times\mathrm{id}} & R\times R\\
\downarrow{\scriptstyle \mathrm{id}\times\mathrm{id}\times\Delta} & & \Big\downarrow\\
R\times R\times R\times R & & \\
\downarrow{\scriptstyle \mathrm{id}\times\sigma\times\mathrm{id}} & & {\scriptstyle m}\\
R\times R\times R\times R & & \\
\downarrow{\scriptstyle m\times m} & & \Big\downarrow\\
R\times R & \xrightarrow[\mu]{} & R
\end{array}$$

We say that R is *commutative* if (R, m, e) is a commutative monoid.

A *morphism of ring objects* is a morphism of both underlying monoids. We denote by $\mathbf{Ring}(\mathcal{C})$ the category of ring objects in $\mathcal{C}$.

Definition 1.31. Let $\mathcal{C}$ be a cartesian category. A *graded ring object* in $\mathcal{C}$ is a sequence of abelian groups $R = \{(R_i, \mu_i, \eta_i)\}_{i\in\mathbb{N}}$ in $\mathcal{C}$ together with maps $m_{i,j} : R_i \times R_j \to R_{i+j}$ and $e : \mathbb{1} \to R_0$ such that for all $i, j, k \in \mathbb{N}$ the following diagrams of graded associativity and graded unitality commute,

$$\begin{array}{ccc}
R_i\times R_j\times R_k & \xrightarrow{\mathrm{id}\times m_{j,k}} & R_i\times R_{j+k}\\
\downarrow{\scriptstyle m_{i,j}\times\mathrm{id}} & & \downarrow{\scriptstyle m_{i,j+k}}\\
R_{i+j}\times R_k & \xrightarrow[m_{i+j,k}]{} & R_{i+j+k}
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{1}\times R_j & \xrightarrow{e\times\mathrm{id}} & R_0\times R_j\\
 & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle m_{0,j}}\\
 & & R_j
\end{array}
\qquad
\begin{array}{ccc}
R_i\times R_0 & \xleftarrow{\mathrm{id}\times e} & R_i\times\mathbb{1}\\
\downarrow{\scriptstyle m_{i,0}} & \swarrow{\scriptstyle\rho} & \\
R_i & &
\end{array}$$

and the following distributivity diagrams commute.

$$
\begin{array}{ccc}
R_i \times R_j \times R_j & \xrightarrow{\mathrm{id}\times\mu_j} & R_i \times R_j \\
\downarrow{\scriptstyle \Delta\times\mathrm{id}\times\mathrm{id}} & & \\
R_i \times R_i \times R_j \times R_j & & \downarrow{\scriptstyle m_{i,j}} \\
\downarrow{\scriptstyle \mathrm{id}\times\sigma\times\mathrm{id}} & & \\
R_i \times R_j \times R_i \times R_j & & \\
\downarrow{\scriptstyle m_{i,j}\times m_{i,j}} & & \\
R_{i+j} \times R_{i+j} & \xrightarrow[\mu_{i+j}]{} & R_{i+j}
\end{array}
\qquad
\begin{array}{ccc}
R_i \times R_i \times R_j & \xrightarrow{\mu_i\times\mathrm{id}} & R_i \times R_j \\
\downarrow{\scriptstyle \mathrm{id}\times\mathrm{id}\times\Delta} & & \\
R_i \times R_i \times R_j \times R_j & & \downarrow{\scriptstyle m_{i,j}} \\
\downarrow{\scriptstyle \mathrm{id}\times\sigma\times\mathrm{id}} & & \\
R_i \times R_j \times R_i \times R_j & & \\
\downarrow{\scriptstyle m_{i,j}\times m_{i,j}} & & \\
R_{i+j} \times R_{i+j} & \xrightarrow[\mu_{i+j}]{} & R_{i+j}
\end{array}
$$

We say R is *commutative* if the following diagram commutes, for all $i, j \in \mathbb{N}$.

$$
\begin{array}{ccc}
R_i \times R_j & \xrightarrow{\sigma} & R_j \times R_i \\
& {\scriptstyle m_{i,j}}\searrow \quad \swarrow{\scriptstyle m_{j,i}} & \\
& R_{i+j} &
\end{array}
$$

A sequence of morphisms $f = \{f_i : R_i \to S_i\}_{i\in\mathbb{N}}$ between two graded ring objects (possibly commutative) is a *morphism of graded ring objects* if each f_i is a morphism of abelian group objects and the following diagrams commute, for every $i, j \in \mathbb{N}$.

$$
\begin{array}{ccc}
R_i \times R_j & \xrightarrow{f_i\times f_j} & S_i \times S_j \\
\downarrow{\scriptstyle m_{i,j}} & & \downarrow{\scriptstyle m_{i,j}} \\
R_{i+j} & \xrightarrow[f_{i+j}]{} & S_{i+j}
\end{array}
\qquad
\begin{array}{ccc}
& & R_0 \\
& {\scriptstyle e}\nearrow & \\
\mathbb{1} & & \downarrow{\scriptstyle f_0} \\
& {\scriptstyle e}\searrow & \\
& & S_0
\end{array}
$$

We denote by $\mathbf{GrRing}(\mathcal{C})$ the category of graded ring objects in $\mathcal{C}$.

PROPOSITION 1.32. *Let $F : \mathcal{C} \to \mathcal{D}$ be a cartesian functor between cartesian categories. There is an induced cartesian functor*

$$F : \mathbf{Ab}(\mathcal{C}) \to \mathbf{Ab}(\mathcal{D}).$$

Let $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$ $(\Downarrow\tau)$ *be a monoidal transformation between cartesian functors. There is an induced monoidal transformation between cartesian functors*

$$\mathbf{Ab}(\mathcal{C}) \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{Ab}(\mathcal{D}) \quad (\Downarrow\tau) .$$

There are analogous results for (commutative) ring objects and (commutative) graded ring objects.

Proof. The proof follows the same pattern as that of Proposition 1.13. As an example, we check one part of the statement. Let $A \in \mathbf{Ab}(\mathcal{C})$. By the proposition just quoted, we know that FA is a commutative monoid. As the inverse map we take $F((-)^{-1}) : FA \to FA$. The outer part of the following diagram is left invertibility.

$$\begin{array}{ccccc}
FA & \xrightarrow{\Delta} & FA \times FA & \xrightarrow{F((-)^{-1})\times\mathrm{id}} & FA \times FA \\
 & \searrow^{F\Delta} & \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle\nabla} \\
{\scriptstyle e}\downarrow & & F(A \times A) & \xrightarrow{F((-)^{-1}\times\mathrm{id})} & F(A \times A) \\
 & \searrow^{Fe} & & & \downarrow{\scriptstyle F\mu} \\
\mathbb{1} & \underset{\nabla_0}{\xrightarrow{\cong}} & F\mathbb{1} & \xrightarrow[F\eta]{} & FA
\end{array}$$

The internal parts commute because F preserves finite products, by naturality of ∇ and because $(-)^{-1}$ satisfies left invertibility. □

4. Augmented monoids

Definition 1.33. Let $\mathcal{C}$ be a category with an initial object I. Define the category $\mathcal{C}^{\mathrm{aug}}$ of *augmented objects* of $\mathcal{C}$ to be the overcategory $(\mathcal{C} \downarrow I)$, i.e. the category of objects of $\mathcal{C}$ with a chosen morphism to I, with the obvious morphisms.

Remark 1.34. In $\mathcal{C}^{\mathrm{aug}}$, the object $I \xrightarrow{\mathrm{id}} I$ is a zero object (i.e. both initial and final). The forgetful functor $\mathcal{C}^{\mathrm{aug}} \to \mathcal{C}$ is an isomorphism of categories if and only if $\mathcal{C}$ has a zero object. In particular, there is an isomorphism of categories $(\mathcal{C}^{\mathrm{aug}})^{\mathrm{aug}} \xrightarrow{\cong} \mathcal{C}^{\mathrm{aug}}$.

Note that if $\mathcal{C}$ is a cartesian category then $\mathbb{1}$ is a final object, and $\mathbf{Ab}(\mathcal{C})$ has $\mathbb{1}$ as a zero object. Thus, $\mathbf{Ab}(\mathcal{C})^{\mathrm{aug}} \cong \mathbf{Ab}(\mathcal{C})$.

Let $\mathcal{V}$ be a symmetric monoidal category.

LEMMA 1.35. *Suppose that the monoidal unit $\mathbb{1} \in \mathcal{V}$ is an initial object. There is an induced symmetric monoidal structure on $\mathcal{V}^{\text{aug}}$. Moreover, the isomorphism of categories* $(\mathcal{V}^{\text{aug}})^{\text{aug}} \xrightarrow{\cong} \mathcal{V}^{\text{aug}}$ *is strict symmetric monoidal.*

PROOF. If $A, B \in \mathcal{V}^{\text{aug}}$ with augmentations denoted by ε, then $A \otimes B$ is augmented, with augmentation

$$A \otimes B \xrightarrow{\varepsilon \otimes \varepsilon} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\lambda} \mathbb{1}.$$

The unit is $\mathbb{1}$ augmented by $\text{id} : \mathbb{1} \to \mathbb{1}$, and the associator, unitor and symmetry are similarly induced. The last statement is obvious. □

The following result is analogous to Proposition 1.13, which applies to non-augmented monoids.

PROPOSITION 1.36. *There is a symmetric monoidal category $\mathbf{Mon}(\mathcal{V})^{\text{aug}}$ of augmented monoids. If $F : \mathcal{V} \to \mathcal{W}$ is a normal lax symmetric monoidal functor, then there is an induced normal lax symmetric monoidal functor*

$$F : \mathbf{Mon}(\mathcal{V})^{\text{aug}} \to \mathbf{Mon}(\mathcal{W})^{\text{aug}}$$

which is strong if F is strong.

If τ is a monoidal transformation between normal lax symmetric monoidal functors $F, G : \mathcal{V} \to \mathcal{W}$, there is an induced monoidal transformation

$$\mathbf{Mon}(\mathcal{V})^{\text{aug}} \underset{G}{\overset{F}{\rightrightarrows}} \Downarrow \tau \; \mathbf{Mon}(\mathcal{W})^{\text{aug}} \tag{1.37}$$

Analogous results hold in the case the monoids are commutative.

PROOF. As observed in Remark 1.27, $\mathbf{Mon}(\mathcal{V})$ has $\mathbb{1}$ as its initial object. Recall from Proposition 1.11 the symmetric monoidal structure on $\mathbf{Mon}(\mathcal{V})$: the previous lemma gives the symmetric monoidal structure of $\mathbf{Mon}(\mathcal{V})^{\text{aug}}$.

The rest of the proof is as that of Proposition 1.13. Note that normality of F is needed so that an augmentation of a monoid $A \to \mathbb{1}$ in $\mathcal{V}$ gets mapped to an augmentation $FA \to F\mathbb{1} \xleftarrow{\cong} \mathbb{1}$ of FA in $\mathcal{W}$. □

The next results will be needed later.

LEMMA 1.38. *Suppose that $\mathbb{1} \in \mathcal{V}$ is an initial object. Then the symmetric monoidal categories $\mathbf{Mon}(\mathcal{V})^{\text{aug}}$ and $\mathbf{Mon}(\mathcal{V}^{\text{aug}})$ are isomorphic via a strict symmetric monoidal functor, and similarly in the commutative case.*

PROOF. If $M \in \mathbf{Mon}(\mathcal{V})^{\text{aug}}$, then the augmentation $\varepsilon : M \to \mathbb{1}$ is a morphism of monoids. If $M \in \mathbf{Mon}(\mathcal{V}^{\text{aug}})$, this means that we have an augmentation map $\varepsilon : M \to \mathbb{1}$ which is a morphism of augmented objects. Both of these conditions on ε are the same, namely the commutativity of the following diagram.

$$\begin{array}{ccc} M \otimes M & \xrightarrow{\mu} & M \\ {\scriptstyle \varepsilon\otimes\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ \mathbb{1} \otimes \mathbb{1} & \xrightarrow[\lambda]{} & \mathbb{1} \end{array}$$

□

COROLLARY 1.39. *Let $\mathcal{V}$ be a symmetric monoidal category. Then the symmetric monoidal categories* $\mathbf{CMon}(\mathbf{Mon}(\mathcal{V})^{\text{aug}})^{\text{aug}}$ *and* $\mathbf{CMon}(\mathcal{V})^{\text{aug}}$ *are isomorphic via a strict symmetric monoidal functor.*

PROOF. Applying Lemma 1.38, then Remark 1.35 and the Eckmann-Hilton argument (Proposition 1.14), we obtain the following isomorphisms of categories which are strict symmetric monoidal.

$$\begin{aligned} \mathbf{CMon}(\mathbf{Mon}(\mathcal{V})^{\text{aug}}))^{\text{aug}} &\cong (\mathbf{CMon}(\mathbf{Mon}(\mathcal{V}))^{\text{aug}})^{\text{aug}} \\ &\cong \mathbf{CMon}(\mathbf{Mon}(\mathcal{V}))^{\text{aug}} \\ &\cong \mathbf{CMon}(\mathcal{V})^{\text{aug}} \end{aligned}$$

□

5. Simplicial objects

If $\mathcal{C}$ is any category, we denote by $s\mathcal{C}$ the category of *simplicial objects* in $\mathcal{C}$, i.e. the category of functors $\mathbb{\Delta}^{\text{op}} \to \mathcal{C}$ where $\mathbb{\Delta}$ is the category with objects $[p] = \{0, \ldots, p\}$, $p \in \mathbb{N}$ and morphims order-preserving functions. It is generated by coface and codegeneracy maps. A functor $\mathbb{\Delta} \to \mathcal{C}$ is called a *cosimplicial object* in $\mathcal{C}$.

Let $\mathcal{V}$ be a symmetric monoidal category. Note that if I is any category, then the category of functors $I \to \mathcal{V}$, denoted $\mathcal{V}^I$, is a symmetric monoidal category with tensor product $(X \otimes Y)(i) = X(i) \otimes Y(i)$ and unit object the constant functor at $\mathbb{1}$, denoted $c\mathbb{1}$. In particular, $s\mathcal{V}$ is a symmetric monoidal category.

Moreover, if $F : \mathcal{V} \to \mathcal{W}$ is a monoidal functor (of any possible variant), then there is an induced monoidal functor $F : \mathcal{V}^I \to \mathcal{W}^I$ of the same variant. In particular, there is a monoidal functor $F : s\mathcal{V} \to s\mathcal{W}$ of the same variant.

We now present a couple of easy lemmas we will use below.

LEMMA 1.40. *If $\mathcal{C}$ is a category with an initial object, then there is an isomorphism of categories $s(\mathcal{C}^{\text{aug}}) \cong (s\mathcal{C})^{\text{aug}}$, and if $\mathcal{C} = \mathcal{V}$ then this isomorphism is strict symmetric monoidal.*

Proof. This is immediate from the definitions. □

Lemma 1.41. *There is an isomorphism of categories between* $\mathbf{Mon}(s\mathcal{V})$ *and* $s\mathbf{Mon}(\mathcal{V})$ *which is strict symmetric monoidal. An analogous statement holds for commutative monoids, and for (cocommutative) comonoids.*

If $\mathcal{V}$ *is cartesian, an analogous statement holds for abelian groups, rings or graded rings.*

Proof. Once more, this is a statement that is easily proven by inspection. For monoids, we sketch a different, more conceptual proof. Recall from Proposition 1.15 that we can replace $\mathbf{Mon}(s\mathcal{V})$ by $\mathbf{Lax}(\mathbf{1}, s\mathcal{V})$. By adjunction, the latter is isomorphic to functors $\mathbf{1} \times \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{V}$ which are lax in the first variable. By adjunction again, this is isomorphic to $s\mathbf{Mon}(\mathcal{V})$. A completely analogous proof can be made for commutative monoids and (cocommutative) comonoids. A similar proof can be made in the cartesian case. This time we do not consider some kind of monoidal functor out of $\mathbf{1}$, but cartesian functors out of the syntactic category associated to the theory; see [**Bor94**, Chapter 3]. □

6. Bimonoids and Hopf monoids

Let $\mathcal{V}$ be a symmetric monoidal category. The material in this section is taken from [**AM10**, 1.2].

Proposition 1.42. *Let* $A \in \mathcal{V}$, $\mu : A \otimes A \to A$, $\eta : \mathbb{1} \to A$, $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{1}$. *The following are equivalent:*

1. $(A, \mu, \eta, \Delta, \varepsilon) \in \mathbf{Comon}(\mathbf{Mon}(\mathcal{V}))$,
2. $(A, \Delta, \varepsilon, \mu, \eta) \in \mathbf{Mon}(\mathbf{Comon}(\mathcal{V}))$,
3. (A, μ, η) *is a monoid,* (A, Δ, ε) *is a comonoid, and the following compatibility diagrams commute.*

$$\begin{array}{ccccc} A\otimes A & \xrightarrow{\mu} & A & \xrightarrow{\Delta} & A\otimes A \\ {\scriptstyle \Delta\otimes\Delta}\downarrow & & & & \uparrow{\scriptstyle \mu\otimes\mu} \\ A\otimes A\otimes A\otimes A & & \xrightarrow[\mathrm{id}\otimes\sigma\otimes\mathrm{id}]{} & & A\otimes A\otimes A\otimes A \end{array}$$

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle \varepsilon\otimes\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ \mathbb{1}\otimes\mathbb{1} & \xrightarrow[\lambda]{} & \mathbb{1} \end{array} \qquad \begin{array}{ccc} \mathbb{1} & \xrightarrow{\lambda^{-1}} & \mathbb{1}\otimes\mathbb{1} \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta\otimes\eta} \\ A & \xrightarrow[\Delta]{} & A\otimes A \end{array} \qquad \begin{array}{ccc} \mathbb{1} & \xrightarrow{\eta} & A \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \varepsilon} \\ & & \mathbb{1} \end{array}$$

Moreover, the properties of commutativity or cocommutativity of A hold in one of the items if and only if they hold in all of them.

Proof. It is just a matter of displaying all the diagrams in the definitions involved. □

Definition 1.43. In the situation of the previous proposition, we say that A is a *bimonoid*. It is *commutative* or *cocommutative* if its underlying monoid or comonoid is commutative or cocommutative. We say it is *bicommutative* if it is both commutative and cocommutative. An arrow $f : A \to B$ between bimonoids, possibly (co/bi)commutative, is a *morphism of bimonoids* if it is a morphism of underlying monoids and comonoids.

If moreover there is a map $s : A \to A$, called the *antipode*, such that the following diagrams commute, then we say that A is a *Hopf monoid*.

$$
\begin{array}{ccccc}
A & \xrightarrow{\Delta} & A\otimes A & \xrightarrow{\mathrm{id}\otimes s} & A\otimes A \\
{\scriptstyle e}\downarrow & & & & \downarrow{\scriptstyle \mu} \\
\mathbb{1} & & \xrightarrow[\eta]{\hspace{4em}} & & A
\end{array}
\qquad
\begin{array}{ccccc}
A & \xrightarrow{\Delta} & A\otimes A & \xrightarrow{s\otimes \mathrm{id}} & A\otimes A \\
{\scriptstyle e}\downarrow & & & & \downarrow{\scriptstyle \mu} \\
\mathbb{1} & & \xrightarrow[\eta]{\hspace{4em}} & & A
\end{array}
\tag{1.44}
$$

A *morphism of Hopf monoids* (possibly bicommutative) is an arrow of the underlying bimonoids. We denote by $\mathbf{BiHopf}(\mathcal{V})$ the symmetric monoidal category of bicommutative Hopf monoids. We will focus on this case since it is the one appearing most often below.

Remark 1.45. As a corollary of Proposition 1.42, one deduces that there is a forgetful functor $\mathbf{BiHopf}(\mathcal{V}) \to \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}}$, where the counit of a Hopf monoid is reinterpreted as an augmentation of a monoid.

In [**AM10**, 1.16], the authors prove that if $f : A \to B$ is a morphism of Hopf monoids, then it commutes with the antipodes.

In Proposition 1.25 we saw that cocommutative comonoids in a symmetric monoidal category are a cartesian category. Therefore, we can internalize groups in it (Definition 1.28).

Proposition 1.46. *Let* $A \in \mathcal{V}$, $\mu : A \otimes A \to A$, $\eta : \mathbb{1} \to A$, $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{1}$, $s : A \to A$. *The following are equivalent:*

(1) $(A, \mu, \eta, \Delta, \varepsilon, s)$ *is a bicommutative Hopf monoid,*
(2) $(A, \Delta, \varepsilon, \mu, \eta, s)$ *is an abelian group object in* $\mathbf{CoComon}(\mathcal{V})$.

Therefore, the symmetric monoidal categories $\mathbf{Ab}(\mathbf{CoComon}(\mathcal{V}))$ *and* $\mathbf{BiHopf}(\mathcal{V})$ *are isomorphic via a strict symmetric monoidal functor.*

Proof. First of all, A is a bicommutative bimonoid if and only if A is a commutative monoid object in $\mathbf{CoComon}(\mathcal{V})$: this follows from Proposition 1.42. The only remaining thing to check is that s is an antipode (commutativity of Diagram 1.44) if and only if it is an inverse for the monoid structure (commutativity of Diagram 1.29). But those diagrams coincide. □

7. Modules and algebras

Let $\mathcal{V}$ be a symmetric monoidal category.

Definition 1.47. Let A be a monoid in $\mathcal{V}$. A *A-module* is an object $M \in \mathcal{V}$ together with an *action* morphism $\rho : A \otimes M \to M$ such that the following diagrams commute.

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mathrm{id}\otimes\rho} & A \otimes M \\ {\scriptstyle \mu\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \rho} \\ A \otimes M & \xrightarrow[\rho]{} & M \end{array} \qquad \begin{array}{ccc} \mathbb{1} \otimes M & \xrightarrow{\eta\otimes\mathrm{id}} & A \otimes M \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle \rho} \\ & & M \end{array}$$

An arrow $f : M \to N$ between left A-modules is a *morphism of modules* if the following diagram commutes.

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\mathrm{id}\otimes f} & A \otimes N \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

We denote by A-**Mod** the category of left A-modules. Similarly, we can define *right A-modules*: we denote the category they form by **Mod**-A. There is also a category of bimodules A-**Bimod**: an *A-bimodule* is an object $M \in \mathcal{V}$ together with a left action ρ of A on M and a right action r of A on M such that the following diagram commutes.

$$\begin{array}{ccc} A \otimes M \otimes A & \xrightarrow{\mathrm{id}\otimes r} & A \otimes M \\ {\scriptstyle \rho\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \rho} \\ M \otimes A & \xrightarrow[r]{} & M \end{array}$$

We say that a bimodule is *symmetric* if the following diagram commutes.

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\rho} & M \\ {\scriptstyle \sigma}\downarrow & \nearrow{\scriptstyle r} & \\ M \otimes A & & \end{array}$$

A *morphism of bimodules* is a morphism of underlying left and right modules.

Remark 1.48. Let $f : A \to B$ be a morphism of monoids in $\mathcal{V}$. It induces a functor $f^* : B$-**Mod** $\to A$-**Mod** of *restriction of scalars* (and similarly for right modules and bimodules): if M is a B-module, we can give its underlying $\mathcal{V}$-object the structure of an A-module with action

$$A \otimes M \xrightarrow{f \otimes \mathrm{id}} B \otimes M \xrightarrow{\rho} M.$$

The axioms of an A-module are easy to check, and the action of f^* on arrows is straightforwardly defined.

Recall from Definition 1.12 the notions of opposite and enveloping monoid. The following lemma is proven by a straightforward verification.

LEMMA 1.49. *The categories* A-**Mod** *and* **Mod**-A^{op} *are isomorphic. Similarly, the categories* **Mod**-A *and* A^{op}-**Mod** *are isomorphic.*

The categories A-**Bimod**, A^e-**Mod** *and* **Mod**-A^e *are isomorphic.*

If $\mathcal{V}$ has coequalizers, we define a pairing

$$- \otimes_A - : \mathbf{Mod}\text{-}A \times A\text{-}\mathbf{Mod} \to \mathcal{V}$$

by the coequalizer in $\mathcal{V}$

$$M \otimes_A N = \mathrm{coeq}\big(M \otimes A \otimes N \underset{\mathrm{id} \otimes \rho_N}{\overset{\rho_M \otimes \mathrm{id}}{\rightrightarrows}} M \otimes N\big).$$

PROPOSITION 1.50. *If* A *is a commutative monoid in* $\mathcal{V}$, $\mathcal{V}$ *has coequalizers and* $A \otimes - : \mathcal{V} \to \mathcal{V}$ *preserves them, then* $(A\text{-}\mathbf{Mod}, \otimes_A, A)$ *is a symmetric monoidal category.*

PROOF. Let $M, N \in A$-**Mod**. Since A is commutative, we can endow M with the right A-module structure given by

$$M \otimes A \xrightarrow{\sigma} A \otimes M \xrightarrow{\rho} M;$$

denote this composition by $\overline{\rho}$. Therefore the expression $M \otimes_A N$ makes sense. (Note that we have actually given M the structure of an A-bimodule.) Now we need to explain how $M \otimes_A N$ is a left A-module. We will denote by $\widehat{\rho}$ the action of A on $M \otimes_A N$.

Since $A \otimes -$ preserves coequalizers, in order to define a candidate left action of A on $M \otimes_A N$ it suffices to define an arrow f commuting with the two following parallel arrows, where $\varphi : M \otimes N \to M \otimes_A N$ is the canonical map.

$$\begin{array}{ccccc} A \otimes M \otimes A \otimes N & \underset{\mathrm{id} \otimes \mathrm{id} \otimes \rho}{\overset{\mathrm{id} \otimes \rho \otimes \mathrm{id}}{\rightrightarrows}} & A \otimes M \otimes N & \xrightarrow{\mathrm{id} \otimes \varphi} & A \otimes (M \otimes_A N) \\ & & & \searrow^{f} & \vdots\, \widehat{\rho} \\ & & & & M \otimes_A N \end{array}$$

Define f as

$$A \otimes M \otimes N \xrightarrow{\sigma \otimes \mathrm{id}} M \otimes A \otimes N \underset{\mathrm{id} \otimes \rho}{\overset{\overline{\rho} \otimes \mathrm{id}}{\rightrightarrows}} M \otimes N \xrightarrow{\varphi} M \otimes_A N.$$

It is easy to verify that this defines indeed an action of A on $M \otimes_A N$, and that A-**Mod** becomes a symmetric monoidal category with unit A and structure morphisms induced by the ones of $\mathcal{V}$. □

REMARK 1.51. A symmetric monoidal category $\mathcal{V}$ is *closed* if for every object $A \in \mathcal{V}$, the functor $A \otimes - : \mathcal{V} \to \mathcal{V}$ has a right adjoint, called an *internal hom* and denoted $\underline{\mathcal{V}}(A, -)$. The hypotheses for the previous proposition are the weakest we need. In practice, it often happens that these hypotheses are guaranteed by the scenario where $\mathcal{V}$ is cocomplete and closed.

The following proposition is not surprising: however, we have not even seen it stated in the literature.

PROPOSITION 1.52. *Let $F : \mathcal{V} \to \mathcal{W}$ be a lax symmetric monoidal functor between symmetric monoidal categories. Let $A \in$ **CMon**$(\mathcal{V})$. Suppose $\mathcal{V}$ has coequalizers and $A \otimes - : \mathcal{V} \to \mathcal{V}$ preserves them, and $\mathcal{W}$ has coequalizers and $FA \otimes - : \mathcal{W} \to \mathcal{W}$ preserves them. Suppose moreover that F preserves coequalizers. There is an induced lax symmetric monoidal functor*

$$F : A\text{-}\mathbf{Mod} \to FA\text{-}\mathbf{Mod}$$

which is strong if F is strong.

PROOF. Let $M \in A$-**Mod**. The FA-module structure of FM is given by

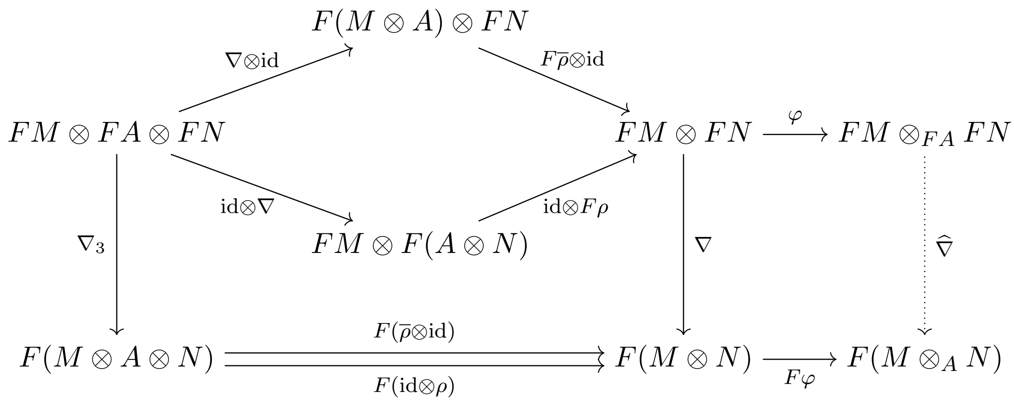
$$FA \otimes FM \xrightarrow{\nabla} F(A \otimes M) \xrightarrow{F\rho} FM.$$

The unit morphism of F is to be $\mathrm{id} : FA \to FA$. We now need to define a morphism of FA-modules

$$\widehat{\nabla} : FM \otimes_{FA} FN \to F(M \otimes_A N)$$

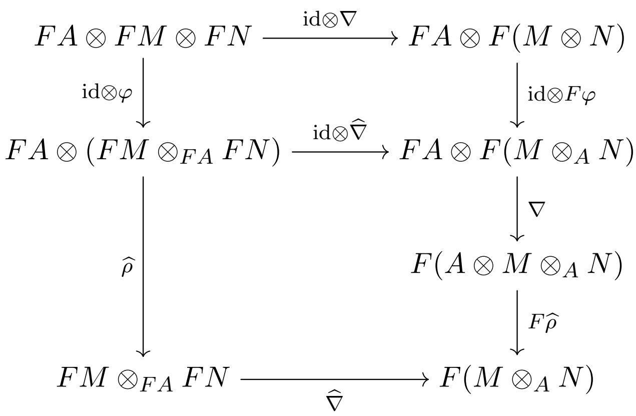
given A-modules M and N. It will be the induced map on coequalizers displayed in the following diagram: here we use that F preserves coequalizers. It is an isomorphism if ∇

is an isomorphism, proving the last claim.

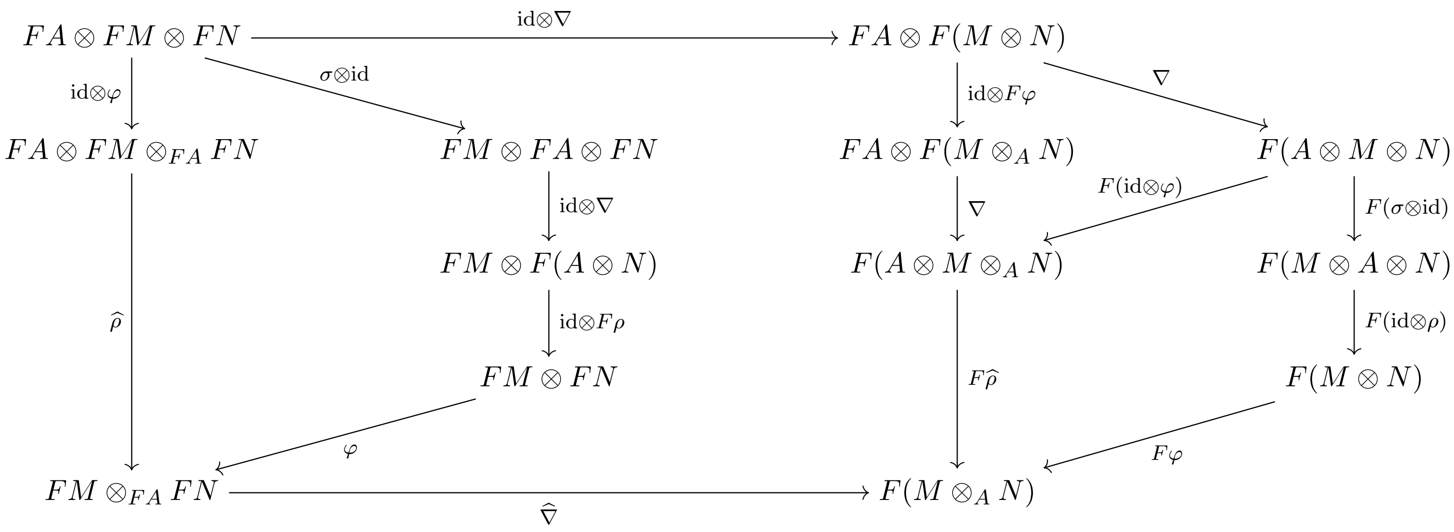


Here the morphisms $\overline{\rho}$ and φ, as well as the morphism $\widehat{\rho}$ of the diagrams to come, were introduced in the proof of Proposition 1.50.

To check that $\widehat{\nabla}$ is a morphism of FA-modules is to check that the lower rectangle in the following diagram commutes. It suffices to check that the big one commutes, since the upper one does and the coequalizer map $\mathrm{id} \otimes \varphi$ is an epimorphism.



This follows from the commutativity of the following diagram.



$FA \otimes FM \otimes FN$
$\mathrm{id}\otimes\nabla$
$FA \otimes F(M \otimes N)$
$\mathrm{id}\otimes\varphi$
$\sigma\otimes\mathrm{id}$
$\mathrm{id}\otimes F\varphi$
∇
$FA \otimes FM \otimes_{FA} FN$
$FM \otimes FA \otimes FN$
$FA \otimes F(M \otimes_A N)$
$F(A \otimes M \otimes N)$
$\mathrm{id}\otimes\nabla$
∇
$F(\mathrm{id}\otimes\varphi)$
$F(\sigma\otimes\mathrm{id})$
$FM \otimes F(A \otimes N)$
$F(A \otimes M \otimes_A N)$
$F(M \otimes A \otimes N)$
$\widehat{\rho}$
$\mathrm{id}\otimes F\rho$
$F(\mathrm{id}\otimes\rho)$
$FM \otimes FN$
$F\widehat{\rho}$
$F(M \otimes N)$
φ
$F\varphi$
$FM \otimes_{FA} FN$
$\widehat{\nabla}$
$F(M \otimes_A N)$

First, the left part commutes by applying $FA \otimes -$ to the coequalizer diagram defining the action of FA on $FM \otimes_{FA} FN$. Second, the right part commutes by naturality of ∇ and by applying F to the coequalizer diagram defining the action of A on $M \otimes_A N$. Thus, it suffices to see that the diagram resulting from deleting the two long vertical compositions from it commutes. This is the commutativity of the following diagram.

$$
\begin{array}{ccc}
FA \otimes FM \otimes FN & \xrightarrow{\mathrm{id}\otimes\nabla} & FA \otimes F(M \otimes N) \\
\downarrow{\scriptstyle \sigma\otimes\mathrm{id}} & & \downarrow{\scriptstyle \nabla} \\
FM \otimes FA \otimes FN & & F(A \otimes M \otimes N) \\
\downarrow{\scriptstyle \mathrm{id}\otimes\nabla} & & \downarrow{\scriptstyle F(\sigma\otimes\mathrm{id})} \\
FM \otimes F(A \otimes N) & \xrightarrow{\nabla} & F(M \otimes A \otimes N) \\
\downarrow{\scriptstyle \mathrm{id}\otimes F\rho} & & \downarrow{\scriptstyle F(\mathrm{id}\otimes\rho)} \\
FM \otimes FN & \xrightarrow{\nabla} & F(M \otimes N) \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle F\varphi} \\
FM \otimes_{FA} FN & \xrightarrow[\widehat{\nabla}]{} & F(M \otimes_A N)
\end{array}
$$

This last diagram is commutative by naturality of ∇ and by definition of $\widehat{\nabla}$.

One can similarly check that $\widehat{\nabla}$ is associative, unital and symmetric. $\square$

Remark 1.53. The hypotheses of the previous proposition are weak. In practice, it generally happens they are implied by $\mathcal{V}$ and $\mathcal{W}$ being closed and cocomplete, and F being a left adjoint.

Definition 1.54. Let A be a commutative monoid in a symmetric monoidal category $\mathcal{V}$ which has coequalizers and such that $A \otimes - : \mathcal{V} \to \mathcal{V}$ preserves them. A *(commutative) A-algebra* is a (commutative) monoid in $(A\text{-}\mathbf{Mod}, \otimes_A, A)$. We denote by $(A\text{-}\mathbf{CAlg}, \otimes_A, A)$ the symmetric monoidal category of commutative A-algebras.

Corollary 1.55. *Let $F : \mathcal{V} \to \mathcal{W}$ and $A \in \mathbf{CMon}(\mathcal{V})$ be as in the hypotheses of Proposition 1.52. There is an induced lax symmetric monoidal functor*

$$F : A\text{-}\mathbf{CAlg} \to FA\text{-}\mathbf{CAlg}$$

which is strong if F is strong.

Proof. This is an application of Proposition 1.13 to the lax symmetric monoidal functor $F : A\text{-}\mathbf{Mod} \to FA\text{-}\mathbf{Mod}$ of Proposition 1.52. $\square$

PROPOSITION 1.56. *Let $F : \mathcal{V} \to \mathcal{W}$ and $A \in \mathbf{CMon}(\mathcal{V})$ be as in the hypotheses of Proposition 1.52. There is a natural transformation*

$$\begin{array}{ccc} \mathbf{CMon}(\mathcal{V}) & \xrightarrow{A\otimes -} & A\text{-}\mathbf{CAlg} \\ {\scriptstyle F}\downarrow & \Nearrow & \downarrow{\scriptstyle F} \\ \mathbf{CMon}(\mathcal{W}) & \xrightarrow[FA\otimes -]{} & FA\text{-}\mathbf{CAlg} \end{array} \tag{1.57}$$

which is an isomorphism when F is strong.

PROOF. First, note that there is a strong symmetric monoidal functor

$$A \otimes - : \mathcal{V} \to A\text{-}\mathbf{Mod}.$$

Indeed, if $B \in \mathcal{V}$, then $A \otimes B \in A\text{-}\mathbf{Mod}$, with action given by $A \otimes A \otimes B \xrightarrow{\mu\otimes\mathrm{id}} A \otimes B$. The monoidal structure of $A \otimes -$ is given by

$$(A \otimes B) \otimes_A (A \otimes B') \xrightarrow{\cong} A \otimes B \otimes B' \qquad \text{and} \qquad A \xrightarrow{\cong} A \otimes \mathbb{1}.$$

Therefore, $A \otimes -$ induces the functor at the top of diagram (1.57), and similarly for the one in the bottom, by Proposition 1.13. The required morphism

$$FA \otimes FB \to F(A \otimes B)$$

natural in $B \in \mathbf{CMon}(\mathcal{V})$ is given by ∇, the structure morphism of F. The only thing one needs to check is that it is a map of FA-commutative algebras: it only remains to check it is a morphism of FA-modules. This is the commutativity of the following outer diagram: the inner diagram commutes by naturality of ∇ and associativity.

$$\begin{array}{ccc} FA \otimes FA \otimes FB & \xrightarrow{\mathrm{id}\otimes\nabla} & FA \otimes F(A \otimes B) \\ {\scriptstyle \nabla\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \nabla} \\ F(A \otimes A) \otimes FB & \xrightarrow{\nabla} & F(A \otimes A \otimes B) \\ {\scriptstyle F\mu\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle F(\mu\otimes\mathrm{id})} \\ FA \otimes FB & \xrightarrow[\nabla]{} & F(A \otimes B) \end{array}$$

□

CHAPTER 2

Simplicial bar constructions

Let $\mathcal{V}$ be a symmetric monoidal category. We will now introduce different versions of simplicial bar constructions and relate them. The examples include classical and topological Hochschild homology and classifying spaces of groups: we will expose them in Chapter 5.

1. Two-sided bar construction

DEFINITION 2.1. [**May72**, Chapter 10]. Let $A \in \mathbf{Mon}(\mathcal{V})$ with multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{1} \to A$, let (M, ρ_M) be a right A-module and (N, ρ_N) be a left A-module. The *simplicial two-sided bar construction* is the simplicial object

$$B_\bullet(M, A, N) \in s\mathcal{V}$$

defined as follows. We have

$$B_n(M, A, N) = M \otimes A^{\otimes n} \otimes N$$

where $A^{\otimes 0}$ means $\mathbb{1}$. The faces $d_i : M \otimes A^{\otimes n} \otimes N \to M \otimes A^{\otimes n-1} \otimes N$, $i = 0, \ldots, n$ are defined as

$$\begin{aligned} d_0 &= \rho_M \otimes \mathrm{id}^{\otimes n-1} \otimes \mathrm{id}, \\ d_i &= \mathrm{id} \otimes \mathrm{id}^{\otimes i-1} \otimes \mu \otimes \mathrm{id}^{\otimes n-i-1} \otimes \mathrm{id} \quad \text{ if } i = 1, \ldots, n-1, \\ d_n &= \mathrm{id} \otimes \mathrm{id}^{\otimes n-1} \otimes \rho_N, \end{aligned}$$

and the degeneracies $s_i : M \otimes A^{\otimes n} \otimes N \to M \otimes A^{\otimes n+1} \otimes N$ are

$$s_i = \mathrm{id} \otimes \mathrm{id}^{\otimes i} \otimes \eta \otimes \mathrm{id}^{\otimes n-i} \otimes \mathrm{id} \qquad \text{for all } i = 0, \ldots, n.$$

This construction is functorial: let $\mathcal{C}$ be the category whose objects are triples (M, A, N) as above, and where a map $(M, A, N) \to (M', A', N')$ is a triple (φ, f, ψ). Here $f : A \to A'$ is a morphism of monoids, and $\varphi : M \to M'$ and $\psi : N \to N'$ are equivariant with respect to f. Then there is a functor $B_\bullet(-, -, -) : \mathcal{C} \to s\mathcal{V}$.

REMARK 2.2. A more general incarnation of the two-sided bar construction appears in [**EKMM97**, XII.1.1]: given a category $\mathcal{C}$, a monad T, a T-algebra C and a right T-functor F, we can build a simplicial object $B_\bullet(F, T, C)$ in $\mathcal{C}$. We will not need this level of generality.

2. Reduced bar construction

DEFINITION 2.3. The *simplicial (reduced) bar construction* in $\mathbf{Mon}(\mathcal{V})^{\mathrm{aug}}$ is the functor

$$B_\bullet : \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}} \to s\mathcal{V}$$

defined as follows. If $A \in \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}}$ with multiplication $\mu : A \otimes A \to A$, unit $\eta : \mathbb{1} \to A$ and augmentation $\varepsilon : A \to \mathbb{1}$, then

$$B_n(A) = A^{\otimes n},$$

where $A^{\otimes 0}$ means $\mathbb{1}$. The faces $d_i : A^{\otimes n} \to A^{\otimes n-1}$, $i = 0, \dots, n$ are defined as

$$d_0 = \varepsilon \otimes \mathrm{id}^{\otimes n-1},$$
$$d_i = \mathrm{id}^{\otimes i-1} \otimes \mu \otimes \mathrm{id}^{\otimes n-i-1} \quad \text{if } i = 1, \dots, n-1,$$
$$d_n = \mathrm{id}^{\otimes n-1} \otimes \varepsilon,$$

and the degeneracies $s_i : A^{\otimes n} \to A^{\otimes n+1}$ are

$$s_i = \mathrm{id}^{\otimes i} \otimes \eta \otimes \mathrm{id}^{\otimes n-i} \qquad \text{for all } i = 0, \dots, n.$$

The action of $B_\bullet$ on arrows is straightforwardly defined.

REMARK 2.4. This bar construction is a particular case of the two-sided bar construction (Definition 2.1): it is naturally isomorphic to $B_\bullet(\mathbb{1}, -, \mathbb{1})$ where if $A \in \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}}$ then $\mathbb{1}$ is viewed as a left and right A-module via the augmentation $\varepsilon : A \to \mathbb{1}$ (Remark 1.48).

PROPOSITION 2.5. *The functor $B_\bullet : \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}} \to s\mathcal{V}$ is strong symmetric monoidal.*

PROOF. Let $A, A' \in \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}}$. Consider the isomorphism

$$A^{\otimes n} \otimes A'^{\otimes n} \xrightarrow{\cong} (A \otimes A')^{\otimes n} \tag{2.6}$$

defined via associators and symmetries, arranged according to the (n, n)-shuffle given by the formula

$$i \mapsto \begin{cases} 2i - 1 & \text{if } i \in \{1, \dots, n\} \\ 2(i - n) & \text{if } i \in \{n+1, \dots, 2n\}. \end{cases}$$

It is the only (n, n)-shuffle that never has two numbers from $\{1, \dots, n\}$ go to consecutive numbers. In other words, the copy of A at the i-th place and the copy of A' at the $(n + i)$-th place are tensored and placed at the i-th place.

This isomorphism is readily seen to commute with faces and degeneracies, thus furnishing the desired natural isomorphism

$$B_\bullet A \otimes B_\bullet A' \xrightarrow{\cong} B_\bullet(A \otimes A') \ .$$

The unit isomorphism $c\mathbb{1} \cong B_\bullet \mathbb{1}$ is the one defined by unitors, which is unique by coherence. Coherence also guarantees that $B_\bullet$ is associative, unital and symmetric, since the isomorphism (2.6) and the unit are defined via associators, symmetries and unitors. □

Proposition 2.7. *Let $F : \mathcal{V} \to \mathcal{W}$ be a normal lax symmetric monoidal functor between symmetric monoidal categories. Then there is a monoidal transformation*

$$\begin{array}{ccc} \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}} & \xrightarrow{B_\bullet} & s\mathcal{V} \\ {\scriptstyle F}\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle F} \\ \mathbf{Mon}(\mathcal{W})^{\mathrm{aug}} & \xrightarrow[B_\bullet]{} & s\mathcal{W} \end{array}$$

which is a natural isomorphism when F is strong.

Proof. Since F is normal lax symmetric monoidal, it induces such a functor between augmented monoids (Proposition 1.36) and between simplicial objects (Chapter 1, Section 5).

Gathering the monoidal transformations of Lemma 1.5 together for all n, we obtain a monoidal transformation

$$\mathcal{V} \;\underset{\coprod_{n\geq 0} F((-)^{\otimes n})}{\overset{\coprod_{n\geq 0} F(-)^{\otimes n}}{\rightrightarrows}}\; \mathrm{Fun}(\mathbb{N}, \mathcal{W}) \qquad \Downarrow \nabla$$

where $\mathbb{N}$ is the discrete category on the natural numbers.

All there is left is to prove is that, when we start from $\mathbf{Mon}(\mathcal{V})^{\mathrm{aug}}$, the components of ∇ are really morphisms in $s\mathcal{W} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{W})$, i.e. that they are compatible with the faces and degeneracies of the simplicial bar construction.

Let $A \in \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}}$ with multiplication $\mu : A \otimes A \to A$. The face map $d_1 : B_3(A) \to B_2(A)$ is

$$\mu \otimes \mathrm{id} : A \otimes A \otimes A \to A \otimes A,$$

and the face map $d_1 : B_3(FA) \to B_2(FA)$ is the composition of the two vertical maps on the left of the following diagram, whose commutativity we need to check.

$$\begin{array}{ccc} FA\otimes FA\otimes FA & \xrightarrow{\nabla_3} & F(A\otimes A\otimes A)\\ {\scriptstyle \nabla\otimes\mathrm{id}}\downarrow & & \\ F(A\otimes A)\otimes FA & & \downarrow{\scriptstyle F(\mu\otimes\mathrm{id})}\\ {\scriptstyle F\mu\otimes\mathrm{id}}\downarrow & & \\ FA\otimes FA & \xrightarrow[\nabla]{} & F(A\otimes A)\end{array}$$

But from coherence considerations we know we can take $\nabla_3 = \nabla \circ (\nabla \otimes \mathrm{id})$, therefore this commutativity is an application of naturality of ∇. All the other face maps $d_i : A^{\otimes n} \to A^{\otimes(n-1)}$ for $i \neq 0, n$ at each level are built in the same fashion, so the proof adapts. For the extremal face maps d_0 and d_n which use the augmentation $\varepsilon : A \to \mathbb{1}$ of A, there is another diagram proof. For example, for $d_0 : A \otimes A \otimes A \to A \otimes A$, the verification is the commutativity of the following outer diagram. The inner diagrams commute by naturality and unitality of ∇.

$$\begin{array}{ccccc} FA\otimes FA\otimes FA & \xrightarrow{\nabla\otimes\mathrm{id}} & F(A\otimes A)\otimes FA & \xrightarrow{\nabla} & F(A\otimes A\otimes A)\\ {\scriptstyle F\varepsilon\otimes\mathrm{id}\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle F(\varepsilon\otimes\mathrm{id})\otimes\mathrm{id}} & & \downarrow{\scriptstyle F(\varepsilon\otimes\mathrm{id}\otimes\mathrm{id})}\\ F\mathbb{1}\otimes FA\otimes FA & \xrightarrow{\nabla\otimes\mathrm{id}} & F(\mathbb{1}\otimes A)\otimes FA & \xrightarrow{\nabla} & F(\mathbb{1}\otimes A\otimes A)\\ {\scriptstyle \nabla_0^{-1}\otimes\mathrm{id}\otimes\mathrm{id}}\downarrow & \swarrow{\scriptstyle F\lambda\otimes\mathrm{id}} & & & \downarrow{\scriptstyle F(\lambda\otimes\mathrm{id})}\\ \mathbb{1}\otimes FA\otimes FA & & & & \\ {\scriptstyle \lambda\otimes\mathrm{id}}\downarrow & & & & \\ FA\otimes FA & & \xrightarrow[\nabla]{} & & F(A\otimes A)\end{array}$$

Compatibility with the degeneracies follows from a similar diagram proof. □

Corollary 2.8. *There is an induced strong symmetric monoidal functor*

$$B_\bullet : \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \to s(\mathbf{CMon}(\mathcal{V})^{\mathrm{aug}}). \tag{2.9}$$

Let $F : \mathcal{V} \to \mathcal{W}$ be a normal lax symmetric monoidal functor between symmetric monoidal categories. There is a monoidal transformation displayed in the following diagram, which is an isomorphism when F is strong.

$$\begin{array}{ccc} \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} & \xrightarrow{B_\bullet} & s(\mathbf{CMon}(\mathcal{V})^{\mathrm{aug}})\\ {\scriptstyle F}\downarrow & \Uparrow & \downarrow{\scriptstyle F}\\ \mathbf{CMon}(\mathcal{W})^{\mathrm{aug}} & \xrightarrow[B_\bullet]{} & s(\mathbf{CMon}(\mathcal{W})^{\mathrm{aug}})\end{array} \tag{2.10}$$

If $\mathcal{C}$ is a cartesian category, there is an induced cartesian functor

$$B_\bullet : \mathbf{Ab}(\mathcal{C}) \to s\mathbf{Ab}(\mathcal{C}),$$

and if $F : \mathcal{C} \to \mathcal{D}$ is a cartesian functor between cartesian categories, then there is a monoidal isomorphism

$$\begin{array}{ccc} \mathbf{Ab}(\mathcal{C}) & \xrightarrow{B_\bullet} & s\mathbf{Ab}(\mathcal{C}) \\ {\scriptstyle F}\big\downarrow & \nearrow\!\!\!\!\nearrow & \big\downarrow{\scriptstyle F} \\ \mathbf{Ab}(\mathcal{D}) & \xrightarrow[B_\bullet]{} & s\mathbf{Ab}(\mathcal{D}). \end{array}$$

Proof. Applying $\mathbf{CMon}(-)^{\mathrm{aug}}$ to $B_\bullet : \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}} \to s\mathcal{V}$ gives a strong symmetric monoidal functor, as per Proposition 1.36. The form (2.9) of $B_\bullet$ is obtained thanks to Corollary 1.39 and Lemmas 1.40 and 1.41. The claim involving diagram (2.10) is again proved using the second part of Proposition 1.36.

If $\mathcal{C}$ is cartesian, then $\mathbf{Ab}(\mathcal{C})$ and $s\mathcal{C}$ are also cartesian, and Proposition 2.5 says that $B_\bullet : \mathbf{Mon}(\mathcal{C})^{\mathrm{aug}} \to s\mathcal{C}$ is cartesian. The proof is now as above, but passing to abelian group objects instead of augmented commutative objects, by means of Proposition 1.32. To finish, recall the isomorphism $\mathbf{Ab}(\mathcal{C})^{\mathrm{aug}} \cong \mathbf{Ab}(\mathcal{C})$ (Remark 1.34). □

Remark 2.11. The symmetric monoidal category $s\mathcal{V}$ itself admits a monoidal bar construction

$$B_\bullet : s\mathbf{Mon}(\mathcal{V})^{\mathrm{aug}} \to s^2\mathcal{V}$$

which is levelwise $B_\bullet : \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}} \to s\mathcal{V}$.

3. Cyclic bar construction

We start by focusing on the case where we do not take coefficients in a bimodule. For that case, see Subsection 3.1. This abstraction of an older construction in homological algebra first appeared under this name in [**Wal79**, 2.3].

Definition 2.12. The *simplicial cyclic bar construction* is a functor

$$B^{\mathrm{cy}}_\bullet : \mathbf{Mon}(\mathcal{V}) \to s\mathcal{V} \tag{2.13}$$

defined as follows. If $A \in \mathbf{Mon}(\mathcal{V})$ with multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{1} \to A$, then

$$B^{\mathrm{cy}}_n(A) = A^{\otimes n+1}.$$

The faces $d_i : A^{\otimes n+1} \to A^{\otimes n}$, $i = 0, \ldots, n$ are defined as

$$d_i = \mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes n-i-1} \quad \text{if } i = 0, \ldots, n-1, \text{ and}$$

$$d_n = (\mu \otimes \mathrm{id}^{\otimes (n-1)}) \circ \sigma_{n+1}$$

where $\sigma_{n+1} : A^{\otimes n+1} \to A^{\otimes n+1}$ is the isomorphism that puts the last A term at the beginning. The degeneracies $s_i : A^{\otimes n+1} \to A^{\otimes n+2}$ are

$$s_i = \mathrm{id}^{\otimes i+1} \otimes \eta \otimes \mathrm{id}^{\otimes n-i} \qquad \text{for all } i = 0, \ldots, n.$$

The action of $B^{\mathrm{cy}}_{\bullet}$ on arrows is straightforwardly defined.

PROPOSITION 2.14. *The functor $B^{\mathrm{cy}}_{\bullet}$ is strong symmetric monoidal.*

PROOF. Analogous to the proof of Proposition 2.5. □

PROPOSITION 2.15. *Let $F : \mathcal{V} \to \mathcal{W}$ be a lax symmetric monoidal functor between symmetric monoidal categories. There is a monoidal transformation*

$$\begin{array}{ccc} \mathbf{Mon}(\mathcal{V}) & \xrightarrow{B^{\mathrm{cy}}_{\bullet}} & s\mathcal{V} \\ {\scriptstyle F}\downarrow & \nearrow\!\!\!\!\!\nearrow & \downarrow{\scriptstyle F} \\ \mathbf{Mon}(\mathcal{W}) & \xrightarrow[B^{\mathrm{cy}}_{\bullet}]{} & s\mathcal{W} \end{array}$$

which is a natural isomorphism when F is strong.

PROOF. Analogous to the proof of Proposition 2.7. □

COROLLARY 2.16. *There is an induced strong symmetric monoidal functor*

$$B^{\mathrm{cy}}_{\bullet} : \mathbf{CMon}(\mathcal{V}) \to s\mathbf{CMon}(\mathcal{V}).$$

Let $F : \mathcal{V} \to \mathcal{W}$ be a lax symmetric monoidal functor between symmetric monoidal categories. There is a monoidal transformation displayed in the following diagram, which is an isomorphism when F is strong.

$$\begin{array}{ccc} \mathbf{CMon}(\mathcal{V}) & \xrightarrow{B^{\mathrm{cy}}_{\bullet}} & s\mathbf{CMon}(\mathcal{V}) \\ {\scriptstyle F}\downarrow & \nearrow\!\!\!\!\!\nearrow & \downarrow{\scriptstyle F} \\ \mathbf{CMon}(\mathcal{W}) & \xrightarrow[B^{\mathrm{cy}}_{\bullet}]{} & s\mathbf{CMon}(\mathcal{W}) \end{array} \tag{2.17}$$

PROOF. This is an application of Proposition 1.13 and of the Eckmann-Hilton argument (Proposition 1.14). □

So far, what we have said for the cyclic construction applies as well to the reduced construction, *mutatis mutandis*. The following is specific to the cyclic case.

For a commutative monoid $A \in \mathbf{CMon}(\mathcal{V})$, we have that $B^{\mathrm{cy}}_{\bullet}(A) \in sA\text{-}\mathbf{CAlg}$. The A-module structure on $A^{\otimes n+1} \cong A \otimes A^{\otimes n}$ is given by acting on the first factor, and the multiplication over A is given by

$$A^{\otimes n+1} \otimes_A A^{\otimes n+1} \cong A \otimes (A^{\otimes n} \otimes A^{\otimes n}) \xrightarrow{\mathrm{id} \otimes \mu} A \otimes A^{\otimes n}$$

where μ denotes the multiplication of $A^{\otimes n} \in \mathbf{CMon}(\mathcal{V})$. This is a particular case of the action described in the proof of Proposition 1.56.

PROPOSITION 2.18. *Let a lax symmetric monoidal functor $F : \mathcal{V} \to \mathcal{W}$ and $A \in \mathbf{CMon}(\mathcal{V})$ be as in the hypotheses of Proposition 1.52. The morphism*

$$B^{\text{cy}}_\bullet(FA) \to F(B^{\text{cy}}_\bullet A) \tag{2.19}$$

of (2.17) is a morphism of simplicial commutative FA-algebras, which is an isomorphism when F is strong.

PROOF. We first explain the algebra structures. The left hand side, $B^{\text{cy}}_\bullet(FA)$, has the structure of simplicial FA-commutative algebra just described. As for the right hand side, by Corollary 1.55, there is an induced functor $F : A\text{-}\mathbf{CAlg} \to FA\text{-}\mathbf{CAlg}$, which induces a functor $F : sA\text{-}\mathbf{CAlg} \to s(FA\text{-}\mathbf{CAlg})$. Since $B^{\text{cy}}_\bullet(A) \in sA\text{-}\mathbf{CAlg}$, we have that $F(B^{\text{cy}}_\bullet A) \in s(FA\text{-}\mathbf{CAlg})$.

It only remains to check that the morphism (2.19) is a morphism of simplicial commutative FA-algebras. This is just like the proof of Proposition 1.56, where instead of ∇ there will be ∇_{n+1} on simplicial level n. □

3.1. Coefficients in a bimodule.

DEFINITION 2.20. Let $A \in \mathbf{Mon}(\mathcal{V})$ with multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{1} \to A$, and let M be an A-bimodule with left action ρ and right action r. The *simplicial cyclic bar construction of A with coefficients in M* is the simplicial object

$$B^{\text{cy}}_\bullet(A, M) \in s\mathcal{V}$$

defined as follows. We have

$$B^{\text{cy}}_n(A, M) = M \otimes A^{\otimes n}.$$

The faces $d_i : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$, $i = 0, \dots, n$ are defined as

$$d_0 = r \otimes \text{id}^{\otimes n-1},$$
$$d_i = \text{id} \otimes \text{id}^{\otimes i-1} \otimes \mu \otimes \text{id}^{\otimes n-i-1} \quad \text{if } i = 1, \dots, n-1, \text{ and}$$
$$d_n = (\rho \otimes \text{id}^{\otimes(n-1)}) \circ \sigma_{n+1}$$

where $\sigma_{n+1} : A^{\otimes n+1} \to A^{\otimes n+1}$ is the isomorphism that puts the last A term at the beginning. The degeneracies $s_i : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n+1}$ are

$$s_i = \text{id} \otimes \text{id}^{\otimes i} \otimes \eta \otimes \text{id}^{\otimes n-i} \qquad \text{for all } i = 0, \dots, n.$$

This construction is functorial: let $\mathcal{C}$ be the category whose objects are pairs (M, A) as above, and where a map $(M, A) \to (M', A')$ is a pair (φ, f). Here $f : A \to A'$ is a morphism of monoids, and $\varphi : M \to M'$ is equivariant with respect to f for both actions. Then there is a functor $B^{\text{cy}}_\bullet(-,-) : \mathcal{C} \to s\mathcal{V}$.

REMARK 2.21. Taking M to be A with both left and right action given by multiplication, then we immediately have that $B^{\text{cy}}_\bullet(A, A) = B^{\text{cy}}_\bullet(A)$ in $s\mathcal{V}$.

In the symmetric case, we can easily relate the cyclic bar construction and the version with coefficients:

PROPOSITION 2.22. *Let $A \in \mathbf{Mon}(\mathcal{V})$ and let M be a symmetric A-bimodule. There is an isomorphism of simplicial $\mathcal{V}$-objects*

$$B^{\text{cy}}_\bullet(A, M) \cong M \otimes_A B^{\text{cy}}_\bullet(A)$$

natural in A and M.

PROOF. The isomorphism in level n is the canonical isomorphism given by $M \otimes A^{\otimes n} \cong M \otimes_A A^{\otimes n+1}$. The symmetry hypothesis is needed for these isomorphisms to commute with the last face map. □

4. Relationship between the notions

We first relate the cyclic bar construction and the two-sided bar construction in two different ways.

PROPOSITION 2.23. *Let $A \in \mathbf{Mon}(\mathcal{V})$ and let M be an A-bimodule. There is an isomorphism of simplicial $\mathcal{V}$-objects*

$$B^{\text{cy}}_\bullet(A, M) \cong M \otimes_{A^e} B_\bullet(A, A, A)$$

natural in A and M.

PROOF. Recall Lemma 1.49: we can see M as a right A^e-module. To explain the left A^e-action on $B_n(A, A, A)$ for any n, first note that if $U \in \mathcal{V}$, then $A \otimes U \otimes A$ can be given an obvious A-bimodule structure (and therefore a left A^e-module structure) by using the multiplication of A. Moreover, $A \otimes U \otimes A \cong A^e \otimes U$ as left A^e-modules.

By applying these considerations to $U = A^{\otimes n}$, we obtain the desired isomorphism at each simplicial level, and it is easily checked that these isomorphisms commute with the faces and degeneracies. □

Let $A \in \mathbf{Mon}(\mathcal{V})$, let M be a right A-module and N be a left A-module. Let us give $M \otimes N$ the A-bimodule structure with left and right actions given as follows, where r denotes the action of A on M and ρ denotes the action of A on N.

$$A \otimes M \otimes N \xrightarrow{\sigma \otimes \text{id}} M \otimes A \otimes N \xrightarrow{\text{id} \otimes \rho} M \otimes N$$

$$M \otimes N \otimes A \xrightarrow{\text{id} \otimes \sigma} M \otimes A \otimes N \xrightarrow{r \otimes \text{id}} M \otimes N$$

PROPOSITION 2.24. *There is an isomorphism of simplicial $\mathcal{V}$-objects*

$$B^{\mathrm{cy}}_{\bullet}(A, M \otimes N) \cong B_{\bullet}(M, A, N)$$

natural in A, M and N.

PROOF. In view of Proposition 2.23, it suffices to establish an isomorphism

$$B_{\bullet}(M, A, N) \cong (M \otimes N) \otimes_{A^e} B_{\bullet}(A, A, A)$$

natural in A, M and N. The proof of this fact is analogous to the one of Proposition 2.23. ☐

COROLLARY 2.25. *Let $A \in \mathbf{Mon}(\mathcal{V})^{\mathrm{aug}}$. There is an isomorphism of simplicial $\mathcal{V}$-objects*

$$B^{\mathrm{cy}}_{\bullet}(A, \mathbb{1}) \cong B_{\bullet}(A)$$

natural in A.

PROOF. Take $M = N = \mathbb{1}$ in the previous proposition and recall Remark 2.4. ☐

CHAPTER 3

Geometric realization

In the previous chapter we introduced simplicial bar constructions, which give simplicial objects in a symmetric monoidal category $\mathcal{V}$. With the geometric realization functors which we will now introduce, we will see how to meaningfully "realize" these simplicial objects as objects of $\mathcal{V}$. Since we want them to have extra structure, we need our geometric realizations to have good monoidal behavior: the main theorems are 3.9 and 3.11.

1. Basic definitions and properties

We now introduce a notion of "intrinsic" geometric realization of a simplicial object in a category $\mathcal{V}$ to an object in $\mathcal{V}$. By the quoted term we mean a functor $s\mathcal{V} \to \mathcal{V}$, in contrast with what happens with the standard geometric realization of a simplicial set into a topological space (see Chapter 5, Section 2), which we call "extrinsic" and which we analyze in Remark 3.5.2. Recall that the geometric realization of a simplicial space can be described as a tensor product of functors [**ML98**, IX.6]: if $X_\bullet$ is a simplicial space, then

$$|X_\bullet| = X_\bullet \otimes_{\mathbb{\Delta}} \Delta^\bullet_{\text{top}} \in \mathbf{Top}$$

where $\Delta^\bullet_{\text{top}} : \mathbb{\Delta} \to \mathbf{Top}$ is the standard cosimplicial space that takes $[n]$ to the geometric n-simplex. This defines a functor $| - | : s\mathbf{Top} \to \mathbf{Top}$.

More generally, we have the following definition.

Definition 3.1. Let $\mathcal{V}$ be a cocomplete symmetric monoidal category with a given cosimplicial object $D^\bullet : \mathbb{\Delta} \to \mathcal{V}$. We define the *geometric realization* functor (with respect to $D^\bullet$) as

$$| - |_{D^\bullet} := - \otimes_{\mathbb{\Delta}} D^\bullet : s\mathcal{V} \to \mathcal{V}. \tag{3.2}$$

explicitly, if $X_\bullet \in s\mathcal{V}$, then $|X_\bullet|_{D^\bullet}$ can be expressed as a coend, or even more explicitly as a coequalizer, as follows:

$$|X_\bullet|_{D^\bullet} = \int^n X_n \otimes D^n = \operatorname{coeq}\left(\coprod_{[n] \xrightarrow{f} [m] \in \operatorname{Arr}\mathbb{\Delta}} X_m \otimes D^n \rightrightarrows \coprod_{[p] \in \mathbb{\Delta}} X_p \otimes D^p \right)$$

where the parallel arrows are defined in the component $[n] \xrightarrow{f} [m]$ of the coproduct as

$$X_m \otimes D^n \xrightarrow{Xf\otimes \mathrm{id}} X_n \otimes D^n \longrightarrow \coprod_{[p]\in\Delta} X_p \otimes D^p \qquad \text{and}$$

$$X_m \otimes D^n \xrightarrow{\mathrm{id}\otimes D^\bullet(f)} X_m \otimes D^m \longrightarrow \coprod_{[p]\in\Delta} X_p \otimes D^p,$$

the maps into the coproduct being the canonical ones.

In the context of this definition, we have:

PROPOSITION 3.3. *If $\mathcal{V}$ is moreover closed, then the functor $|-|_{D^\bullet} : s\mathcal{V} \to \mathcal{V}$ is a left adjoint. The right adjoint is given by the functor*

$$\underline{\mathcal{V}}(D^\bullet, -) : \mathcal{V} \to s\mathcal{V}, \qquad V \mapsto ([n] \mapsto \underline{\mathcal{V}}(D^n, V))$$

where $\underline{\mathcal{V}}(-,-)$ denotes the internal hom object of $\mathcal{V}$.

PROOF. Let $X_\bullet \in s\mathcal{V}$ and $V \in \mathcal{V}$. Then

$$\mathcal{V}(|X_\bullet|_{D^\bullet}, V) = \mathcal{V}\left(\int^n X_n \otimes D^n, V\right) \cong \int_n \mathcal{V}(X_n \otimes D^n, V) \cong$$
$$\cong \int_n \mathcal{V}(X_n, \underline{\mathcal{V}}(D^n, V)) \cong \mathrm{Nat}(X_\bullet, \underline{\mathcal{V}}(D^\bullet, V)) = s\mathcal{V}(X_\bullet, \underline{\mathcal{V}}(D^\bullet, V))$$

where we have used that the representable functor $\mathcal{V}(-, V) : \mathcal{V}^{\mathrm{op}} \to \mathbf{Set}$ takes coends to ends, and that the set $\mathrm{Nat}(F, G)$ of natural transformations from F to G is equal to the end $\int_X \mathcal{V}(FX, GX)$ for functors F and G out of some small category [**ML98**, p. 223]. □

COROLLARY 3.4. *The geometric realization of a simplicial object $cX \in s\mathcal{V}$ constant at $X \in \mathcal{V}$ is isomorphic to $X \otimes D^0$.*

PROOF. Indeed, following the first steps of the previous proof, we have isomorphisms

$$\mathcal{V}(|cX|_{D^\bullet}, V) \cong \mathrm{Nat}(cX, \underline{\mathcal{V}}(D^\bullet, V)) \cong \mathcal{V}(X, \lim \underline{\mathcal{V}}(D^\bullet, V)) \cong \lim \mathcal{V}(X, \underline{\mathcal{V}}(D^\bullet, V)) \cong$$
$$\cong \lim \mathcal{V}(X \otimes D^\bullet, V) \cong \mathcal{V}(\mathrm{colim}(X \otimes D^\bullet), V) \cong \mathcal{V}(X \otimes D^0, V)$$

naturally in $V \in \mathcal{V}$. Yoneda's lemma implies $|cX|_{D^\bullet} \cong X \otimes D^0$. Here we have used the fact that the colimit of a cosimplicial object is its zeroth component, since Δ has $[0]$ as its final object (see e.g. [**ML98**, Exercise 3, p.72] or [**Rie14**, 8.3.1]). □

REMARK 3.5. It is interesting to note that geometric realization can be extended to an enriched context (see [**Kel05**]), in two different ways. Let $\mathcal{C}$ be a cocomplete category enriched, tensored and cotensored over a closed symmetric monoidal category $\mathcal{V}$. Then:

(1) If we are still given a cosimplicial object $D^\bullet : \mathbb{\Delta} \to \mathcal{V}$, then we can define $|-|_{D^\bullet} : s\mathcal{C} \to \mathcal{C}$ as $|X_\bullet|_{D^\bullet} = \int^n X_n \odot D^n$ where $\odot$ denotes the tensoring of $\mathcal{C}$ over $\mathcal{V}$. The proof of Proposition 3.3 adapts to prove that $|-|_{D^\bullet} : s\mathcal{C} \to \mathcal{C}$ is a left adjoint. If we take $\mathcal{C} = \mathcal{V}$, we recover the construction above.
(2) If instead we are given a cosimplicial object $D^\bullet : \mathbb{\Delta} \to \mathcal{C}$, then we can define $|-|^e_{D^\bullet} : s\mathcal{V} \to \mathcal{C}$ as $|X_\bullet|^e_{D^\bullet} = \int^n D^n \odot X_n$. Again, $|-|^e_{D^\bullet}$ is a left adjoint, and taking $\mathcal{C} = \mathcal{V}$ recovers the construction above. Since this functor takes simplicial objects in one category and begets objects in a different category, we call it *extrinsic geometric realization.*

2. Cosimplicial objects induced by the Yoneda embedding

We will concentrate on closed symmetric monoidal categories with a cosimplicial object induced by the Yoneda embedding on simplicial sets. More precisely,

> *From here until the end of the chapter we let $\mathcal{V}$ be a cocomplete, closed symmetric monoidal category together with a lax symmetric monoidal functor $F : s\mathbf{Set} \to \mathcal{V}$ which is a left adjoint. Let $\Delta^\bullet : \mathbb{\Delta} \to s\mathbf{Set}$ be the Yoneda embedding: we consider the cosimplicial object on $\mathcal{V}$ given by $F\Delta^\bullet : \mathbb{\Delta} \to \mathcal{V}$. We denote by $|-|$ the geometric realization functor $|-|_{F\Delta^\bullet} : s\mathcal{V} \to \mathcal{V}$. If $\mathcal{V}$ is cartesian, we replace the letter $\mathcal{V}$ by $\mathcal{C}$.*

This cosimplicial object $F\Delta^\bullet$ induces a geometric realization functor $|-| : s\mathcal{V} \to \mathcal{V}$. While we choose not to add F to the notation, it is important to keep in mind that the definition of $|-|$ depends on the choice of F.

The particularities of the Yoneda embedding will allow us to do things we could not do with the geometric realization with respect to an abstract cosimplicial object (see e.g. formula (3.10)).

REMARK 3.6. Let $X \in \mathcal{V}$. Since Δ^0 is the unit of the cartesian category $s\mathbf{Set}$, the unit of F has the form $\nabla_0 : \mathbb{1} \to F\Delta^0$. Therefore we get an arrow

$$X \xrightarrow[\cong]{\rho^{-1}} X \otimes \mathbb{1} \xrightarrow{\mathrm{id}\otimes\nabla_0} X \otimes F\Delta^0 \cong |cX| \tag{3.7}$$

which is an isomorphism if F is normal. The last isomorphism is provided by Corollary 3.4.

We recall the *density theorem* (see for example [**Rie14**, 1.4.6]): if $H : I^{\mathrm{op}} \to \mathcal{C}$ is a functor from the opposite of a small category I into a cocomplete category $\mathcal{C}$, then

$$Hj \cong \int^i I(j,i) \cdot Hi. \tag{3.8}$$

Here and henceforth $\cdot$ denotes the tensoring of $\mathcal{C}$ over **Set**. Explicitly, if $A \in \mathbf{Set}$ and $C \in \mathcal{C}$, then $A \cdot C$ is the coproduct $\coprod_{i \in A} C$.

Note that since $F : s\mathbf{Set} \to \mathcal{V}$ is a left adjoint it preserves coproducts, and therefore $F(A \cdot X) \cong A \cdot FX$.

THEOREM 3.9. *The geometric realization functor* $|-| : s\mathcal{V} \to \mathcal{V}$ *has a lax symmetric monoidal structure, which is strong (resp. normal) if* F *is strong (resp. normal).*

PROOF. Let us express the simplicial set $\Delta^n \times \Delta^m$ as a coend, using the density theorem.

$$(\Delta^n \times \Delta^m)(j) \cong \int^i \mathbb{\Delta}(j,i) \times (\Delta^n \times \Delta^m)(i) = \int^i \Delta^i(j) \times (\mathbb{\Delta} \times \mathbb{\Delta})((i,i),(n,m))$$

$$= \left(\int^i (\mathbb{\Delta} \times \mathbb{\Delta})((i,i),(n,m)) \cdot \Delta^i \right)(j) \tag{3.10}$$

Let $X_\bullet, Y_\bullet \in s\mathcal{V}$. We will now repeatedly use Fubini's theorem for coends [**ML98**, IX.8], the fact that $- \otimes -$ commutes with colimits (hence with coends) separately in each variable, and the fact that F commutes with coends. Finally, we use the density theorem again.

$$\begin{aligned}
|X_\bullet| \otimes |Y_\bullet| &= \int^n X_n \otimes F\Delta^n \otimes \int^m Y_m \otimes F\Delta^m \\
&\cong \int^{n,m} X_n \otimes Y_m \otimes F\Delta^n \otimes F\Delta^m \\
&\to \int^{n,m} X_n \otimes Y_m \otimes F(\Delta^n \times \Delta^m) \\
&\cong \int^{n,m} X_n \otimes Y_m \otimes F\left(\int^i (\mathbb{\Delta} \times \mathbb{\Delta})((i,i),(n,m)) \cdot \Delta^i \right) \\
&\cong \int^{n,m} X_n \otimes Y_m \otimes \int^i (\mathbb{\Delta} \times \mathbb{\Delta})((i,i),(n,m)) \cdot F\Delta^i \\
&\cong \int^i \left(\int^{n,m} (\mathbb{\Delta} \times \mathbb{\Delta})((i,i),(n,m)) \cdot (X_n \otimes Y_m) \right) \otimes F\Delta^i \\
&\cong \int^i X_i \otimes Y_i \otimes F\Delta^i = |X_\bullet \otimes Y_\bullet|
\end{aligned}$$

The unit of $|-|$ is furnished by (3.7) applied to $X = \mathbb{1}$.

It is a long, if tedious verification that these morphisms endow $|-|$ with the structure of a lax symmetric monoidal functor. $\square$

3. Behavior under monoidal functors

From here until the end of the chapter we let $\mathcal{W}$ *be a cocomplete, closed symmetric monoidal category together with a lax symmetric monoidal* *functor $G : \mathcal{V} \to \mathcal{W}$ which is a left adjoint. We endow $\mathcal{W}$ with the cosimplicial object $GF\Delta^\bullet : \mathbb{\Delta} \to \mathcal{W}$, and we denote by $|-|$ the geometric realization functor $|-|_{GF\Delta^\bullet} : s\mathcal{W} \to \mathcal{W}$. If $\mathcal{W}$ is cartesian, we replace the letter $\mathcal{W}$ by $\mathcal{D}$.*

Of course, Theorem 3.9 applies to $\mathcal{W}$ as well: $|-| : s\mathcal{W} \to \mathcal{W}$ has a lax symmetric monoidal structure which is strong (resp. normal) if F and G are strong (resp. normal).

THEOREM 3.11. *There is a monoidal transformation between lax symmetric monoidal functors*

$$s\mathcal{V} \underset{G|-|}{\overset{|G-|}{\rightrightarrows}} \mathcal{W}, \quad \Downarrow \tau \tag{3.12}$$

which is an isomorphism if G is strong.

In particular, this holds when $\mathcal{V} = s\mathbf{Set}$ and $F = \mathrm{id}_{s\mathbf{Set}}$.

PROOF. Since G is a left adjoint, it preserves coends, and thus we get

$$\int^n GX_n \otimes GF\Delta^n \to \int^n G(X_n \otimes F\Delta^n) \cong G \int^n X_n \otimes F\Delta^n$$

for $X_\bullet \in s\mathcal{V}$, defining the desired natural transformation. We need to check it is monoidal, i.e. that the following diagram commutes, for $X_\bullet, Y_\bullet \in s\mathcal{V}$.

$$\begin{array}{ccccc} |GX_\bullet| \otimes |GY_\bullet| & \longrightarrow & |GX_\bullet \otimes GY_\bullet| & \longrightarrow & |G(X_\bullet \otimes Y_\bullet)| \\ \downarrow{\scriptstyle \tau\otimes\tau} & & & & \downarrow{\scriptstyle \tau} \\ G|X_\bullet| \otimes G|Y_\bullet| & \longrightarrow & G(|X_\bullet| \otimes |Y_\bullet|) & \longrightarrow & G|X_\bullet \otimes Y_\bullet| \end{array}$$

The horizontal arrows intertwine the monoidal structure of G and of the induced functor $G : s\mathcal{V} \to s\mathcal{W}$ with the monoidal structure of the geometric realizations $|-|$ in $s\mathcal{V}$ and in $s\mathcal{W}$ obtained in Theorem 3.9. These structures are defined via a fairly long string of isomorphisms, whence the difficulty of reproducing the necessary diagram proof. One can expand the diagram into one big rectangle filled with coends, and juggle around with naturality and monoidality properties of F and G. The gist of the proof is that the geometric realizations in $\mathcal{V}$ and in $\mathcal{W}$ are not independent: they are related via G, which is lax symmetric monoidal. □

3.1. In monoids. Suppose F and G are normal. By Theorem 3.9, the geometric realization functor $|-| : s\mathcal{V} \to \mathcal{V}$ is normal lax symmetric monoidal, therefore induces a functor

$$|-| : s\mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \to \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \tag{3.13}$$

and similarly for $|-|: s\mathcal{W} \to \mathcal{W}$.

Since the natural transformation of Theorem 3.11 is monoidal, we can apply Proposition 1.36 and obtain a natural transformation

$$s\mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \underset{G|-|}{\overset{|G-|}{\Downarrow \tau}} \mathbf{CMon}(\mathcal{W})^{\mathrm{aug}} \tag{3.14}$$

which is an isomorphism if G is strong.

A completely analogous result holds with non-augmented objects, in which case we do not need F and G to be normal.

Similarly, if $F : s\mathbf{Set} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ are cartesian functors between cartesian categories, we can apply Proposition 1.32 and obtain a natural isomorphism

$$s\mathbf{Ab}(\mathcal{C}) \underset{G|-|}{\overset{|G-|}{\Downarrow \tau}} \mathbf{Ab}(\mathcal{D}) \tag{3.15}$$

and similarly for (commutative) rings and (commutative) graded rings.

4. Realized bar constructions

In Chapter 2 we have defined different simplicial bar constructions: these have simplicial objects as outputs. We will now apply geometric realization to these. We will focus on the cases that will be important for us below. We are still working with the notation introduced at the beginning of Sections 2 and 3.

4.1. Reduced bar construction. Suppose F is normal. We define the *(reduced) bar construction*

$$B : \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \to \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \tag{3.16}$$

as the composite

$$\mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \xrightarrow{B_\bullet} s(\mathbf{CMon}(\mathcal{V}))^{\mathrm{aug}} \xrightarrow{|-|} \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}}$$

by means of Corollary 2.8 and Proposition 1.36 applied to $|-|$. Remark that the monoid structure on BA for $A \in \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}}$ is induced by the simplicial map

$$A^{\otimes p} \otimes A^{\otimes p} \to A^{\otimes p} \tag{3.17}$$

which is the monoid structure on $A^{\otimes p}$.

Similarly, if $\mathcal{V} = \mathcal{C}$ is cartesian and F is cartesian, we can take abelian group objects instead of augmented commutative monoids, and obtain a functor

$$B : \mathbf{Ab}(\mathcal{C}) \to \mathbf{Ab}(\mathcal{C}). \tag{3.18}$$

Proposition 3.19. *Suppose F and G are normal. There is a natural transformation*

$$\begin{array}{ccc} \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} & \xrightarrow{B} & \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \\ {\scriptstyle G}\downarrow & \Uparrow & \downarrow{\scriptstyle G} \\ \mathbf{CMon}(\mathcal{W})^{\mathrm{aug}} & \xrightarrow[B]{} & \mathbf{CMon}(\mathcal{W})^{\mathrm{aug}} \end{array}$$

which is an isomorphism if G is strong.

If $\mathcal{V}$, $\mathcal{W}$, F and G are cartesian, there is an analogous square for $\mathbf{Ab}$ *instead of* $\mathbf{CMon}^{\mathrm{aug}}$.

Proof. The natural transformation is the pasting of the following two.

$$\begin{array}{ccccc} \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} & \xrightarrow{B_\bullet} & s\mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} & \xrightarrow{|-|} & \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \\ {\scriptstyle G}\downarrow & \Uparrow & \downarrow{\scriptstyle G} & \Uparrow & \downarrow{\scriptstyle G} \\ \mathbf{CMon}(\mathcal{W})^{\mathrm{aug}} & \xrightarrow[B_\bullet]{} & s\mathbf{CMon}(\mathcal{W})^{\mathrm{aug}} & \xrightarrow[|-|]{} & \mathbf{CMon}(\mathcal{W})^{\mathrm{aug}} \end{array}$$

The left one comes from Corollary 2.8, and the right one is (3.14). For abelian groups it is entirely analogous. □

Remark 3.20 (Bicommutative Hopf monoids). By Proposition 1.25, the category $\mathbf{CoComon}(\mathcal{V})$ of cocommutative comonoids in $\mathcal{V}$ is cartesian, therefore admits a simplicial bar construction on its abelian group objects (Corollary 2.8). By Proposition 1.46, $\mathbf{Ab}(\mathbf{CoComon}(\mathcal{V}))$ is isomorphic to $\mathbf{BiHopf}(\mathcal{V})$ via a strict symmetric monoidal functor, so the simplicial bar construction in this case is actually a functor $B_\bullet : \mathbf{BiHopf}(\mathcal{V}) \to s(\mathbf{BiHopf}(\mathcal{V}))$. This functor coincides with the simplicial bar construction of Corollary 2.8 under the forgetful functors U from (simplicial) bicommutative Hopf monoids to (simplicial) commutative augmented monoids (recall Remark 1.45).

Suppose $F : s\mathbf{Set} \to \mathcal{V}$ is strong. As seen in Theorem 3.9, this implies that $|-| : s\mathcal{V} \to \mathcal{V}$ is strong symmetric monoidal. By first passing to cocommutative comonoids and then to abelian group objects, we obtain an induced functor $|-| : s\mathbf{BiHopf}(\mathcal{V}) \to \mathbf{BiHopf}(\mathcal{V})$ which coincides with (3.13) under the forgetful functors U.

Composing $B_\bullet$ and $|-|$ we get a functor

$$B : \mathbf{BiHopf}(\mathcal{V}) \to \mathbf{BiHopf}(\mathcal{V}) \tag{3.21}$$

which coincides with (3.16) under U.

4.2. Cyclic bar construction. We define the *cyclic bar construction*

$$B^{\mathrm{cy}} : \mathbf{CMon}(\mathcal{V}) \to \mathbf{CMon}(\mathcal{V})$$

as the composite

$$\mathbf{CMon}(\mathcal{V}) \xrightarrow{B^{\mathrm{cy}}_\bullet} s\mathbf{CMon}(\mathcal{V}) \xrightarrow{|-|} \mathbf{CMon}(\mathcal{V})$$

by means of Corollary 2.16 and Proposition 1.13.

PROPOSITION 3.22. *Let $A \in \mathbf{CMon}(\mathcal{V})$. If F is normal, the geometric realization functor induces a functor*

$$|-| : sA\text{-}\mathbf{CAlg} \to A\text{-}\mathbf{CAlg}.$$

PROOF. Recall that cA denotes the constant simplicial object at A. Then

$$cA \in s\mathbf{CMon}(\mathcal{V}) \cong \mathbf{CMon}(s\mathcal{V})$$

(cf. Lemma 1.41). Applying Corollary 1.55 to $|-| : s\mathcal{V} \to \mathcal{V}$ we get a functor

$$|-| : sA\text{-}\mathbf{CAlg} \cong cA\text{-}\mathbf{CAlg} \to |cA|\text{-}\mathbf{CAlg}.$$

Since F is normal, then $|cA| \cong A$ (Remark 3.6). □

COROLLARY 3.23. *Let $A \in \mathbf{CMon}(\mathcal{V})$. Then $B^{\mathrm{cy}}(A) \in A\text{-}\mathbf{CAlg}$.*

PROOF. Just recall from Chapter 2, Section 3 that $B^{\mathrm{cy}}_\bullet(A) \in sA\text{-}\mathbf{CAlg}$ and apply the previous proposition. □

PROPOSITION 3.24. *There is a natural transformation*

$$\begin{array}{ccc} \mathbf{CMon}(\mathcal{V}) & \xrightarrow{B^{\mathrm{cy}}} & \mathbf{CMon}(\mathcal{V}) \\ {\scriptstyle G}\downarrow & \nearrow\!\!\!\!\nearrow & \downarrow{\scriptstyle G} \\ \mathbf{CMon}(\mathcal{W}) & \xrightarrow[B^{\mathrm{cy}}]{} & \mathbf{CMon}(\mathcal{W}) \end{array}$$

which is an isomorphism if G is strong.

Moreover, if $A \in \mathbf{CMon}(\mathcal{V})$ and F and G are normal, then the morphism $B^{\mathrm{cy}}(GA) \to G(B^{\mathrm{cy}}A)$ is a morphism of commutative GA-algebras.

PROOF. The first claim is proven entirely analogously to Proposition 3.19. For the second one, recall that $B^{\mathrm{cy}}_\bullet(GA) \to G(B^{\mathrm{cy}}_\bullet A)$ is a morphism of simplicial commutative GA-algebras. The previous proposition applies to finish the proof. □

CHAPTER 4

Iterated reduced bar constructions

In the previous chapter we described the "reduced" and the "cyclic" bar constructions: we will now focus on the former, which is an endofunctor on commutative augmented monoids in a symmetric monoidal category. We can iterate these functors to obtain a family indexed on the naturals. We will endow this family with a graded multiplication, provided we start with a ring object in a cartesian category. If we are dealing with symmetric monoidal categories which are not cartesian, this construction can be made to work: see Section 2.

We work in the context summarized at the beginning of Sections 2 and 3 of Chapter 3.

1. Graded multiplication

We can iterate the bar construction $B : \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \to \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}}$ of (3.16) to obtain functors

$$B^n : \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \to \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \quad \text{for } n \geq 0,$$

where we define B^0 to be the identity functor.

If $\mathcal{V} = \mathcal{C}$ is cartesian, we similarly obtain functors

$$B^n : \mathbf{Ab}(\mathcal{C}) \to \mathbf{Ab}(\mathcal{C}) \quad \text{for } n \geq 0.$$

We will now put a graded ring structure on these iterated bar constructions, provided we start with a ring object. To be able to carry this out, we make the following assumptions.

> *We let the monoidal structures in our categories $\mathcal{V}, \mathcal{W}$ be cartesian. We therefore change notation: we are given cartesian categories $\mathcal{C}, \mathcal{D}$, a cartesian functor $F : s\mathbf{Set} \to \mathcal{C}$ and a cartesian functor $G : \mathcal{C} \to \mathcal{D}$, both of which are left adjoints.*

Thus we get induced geometric realizations on $s\mathcal{C}$ and on $s\mathcal{D}$ which are cartesian, by Theorem 3.9.

We can glue all the iterated bar constructions together into a single functor

$$B^* : \mathbf{Ab}(\mathcal{C}) \to \mathbf{GrAb}(\mathcal{C}), \quad A \mapsto (B^n A)_{n \in \mathbb{N}}.$$

Here $\mathbf{GrAb}(\mathcal{C})$ stands for the category of $\mathbb{N}$-graded objects of $\mathbf{Ab}(\mathcal{C})$, i.e. the functor category $\mathrm{Fun}(\mathbb{N}, \mathbf{Ab}(\mathcal{C}))$ where $\mathbb{N}$ is considered as a discrete category.

Recall the category $\mathbf{GrRing}(\mathcal{C})$ of graded ring objects in $\mathcal{C}$ (Definition 1.31).

THEOREM 4.1. *a) The functor B^* extends to a functor*

$$B^* : \mathbf{Ring}(\mathcal{C}) \to \mathbf{GrRing}(\mathcal{C}).$$

b) There is a natural isomorphism

$$\begin{array}{ccc} \mathbf{Ring}(\mathcal{C}) & \xrightarrow{B^*} & \mathbf{GrRing}(\mathcal{C}) \\ {\scriptstyle G}\downarrow & \Nearrow & \downarrow{\scriptstyle G} \\ \mathbf{Ring}(\mathcal{D}) & \xrightarrow[B^*]{} & \mathbf{GrRing}(\mathcal{D}). \end{array}$$

PROOF. a) Let $S \in \mathbf{Ring}(\mathcal{C})$. Denote by $\mu : S \times S \to S$ its multiplication. We define the graded multiplication $\smile_{n,m}: B^n S \times B^m S \to B^{n+m} S$ inductively.

For $n = m = 0$, $\smile_{0,0}: S \times S \to S$ is μ. Now let us define $\smile_{0,m+1}$ from $\smile_{0,m}$.

Consider, for $i = 1, \ldots, p$, the composition

$$S \times (B^m S)^{\times p} \xrightarrow{\mathrm{id}\times\varepsilon_i} S \times B^m S \xrightarrow{\smile_{0,m}} B^m S, \tag{4.2}$$

where ε_i denotes the i-th projection map. By the universal property of the categorical product, these maps define a morphism

$$\varphi_m^p : S \times (B^m S)^{\times p} \to (B^m S)^{\times p}$$

in $\mathcal{C}$ which commutes with the faces and degeneracies of $B_\bullet B^m S$. Indeed, as an example, the commutativity with the face maps $d_1, \ldots, d_{p-1}$ rests on the distributivity of $\smile_{0,m}$ with respect to the addition of $B^m S$, and this is obtained inductively: for $m = 0$ it is the mere distributivity of $\smile_{0,0} = \mu$ with respect to addition. We say more about the distributivity of the higher $\smile_{n,m}$ at the end of the proof.

We thus get a morphism

$$\varphi_m : S \times B_\bullet B^m S \to B_\bullet B^m S$$

in $s\mathcal{C}$, where S is seen as a constant simplicial object.

As the geometric realization of a constant simplicial object gives the original object (Remark 3.6) and as geometric realization is a cartesian functor, we obtain an induced map

$$S \times B^{m+1} S \xrightarrow{\cong} |S \times B_\bullet B^m S| \xrightarrow{|\varphi_m|} B^{m+1} S \tag{4.3}$$

which we call $\smile_{0,m+1}$.

The definition of $\smile_{n+1,m}$ from $\smile_{n,m}$ is symmetrical: replace (4.2) with

$$(B^nS)^{\times p} \times B^mS \xrightarrow{\varepsilon_i \times \mathrm{id}} B^nS \times B^mS \xrightarrow{\smile_{n,m}} B^{n+m}S$$

and repeat the process.

The unit for this graded multiplication is the unit for the multiplication of S. Associativity and distributivity of $\smile$ follow from associativity and distributivity of μ; these are all straightforward verifications. As an example, here is the diagram expressing the distributivity of $\smile_{0,1}$ at the simplicial level,

$$\begin{array}{ccccc}
S \times S^{\times p} \times S^{\times p} & \xrightarrow{\Delta \times \mathrm{id} \times \mathrm{id}} & S \times S \times S^{\times p} \times S^{\times p} & \xrightarrow{\mathrm{id} \times \sigma \times \mathrm{id}} & S \times S^{\times p} \times S \times S^{\times p} \\
\Big\downarrow {\scriptstyle \mathrm{id} \times +} & & & & \Big\downarrow {\scriptstyle \varphi_0^p \times \varphi_0^p} \\
 & & & & S^{\times p} \times S^{\times p} \\
 & & & & \Big\downarrow {\scriptstyle +} \\
S \times S^{\times p} & & \xrightarrow[\varphi_0^p]{\qquad\qquad\qquad} & & S^{\times p}
\end{array}$$

where $+ : S^{\times p} \times S^{\times p} \to S^{\times p}$ is the abelian group structure map of $S^{\times p}$ (3.17), $\Delta : S \to S \times S$ is the diagonal, and σ is the symmetry. Its commutativity follows from distributivity in S.

b) First of all, the vertical functors induced by G do exist since $G : \mathcal{C} \to \mathcal{D}$ is cartesian (Proposition 1.32). Second, the commutativity at the level of each abelian group object follows by iterating Proposition 3.19.

For the compatibility of $\smile$-multiplications, first observe that when $n = m = 0$ this is just the definition of the multiplication in GS, for $S \in \mathbf{Ring}(\mathcal{C})$:

$$\begin{array}{ccc}
GS \times GS & \xrightarrow{\mu^{GS}} & GS \\
{\scriptstyle \cong} \Big\downarrow & & \Big\| \\
G(S \times S) & \xrightarrow[G\mu^S]{} & GS.
\end{array}$$

For general n and m, this amounts to the commutativity of the following diagram in $\mathcal{D}$,

$$\begin{array}{ccc}
B^nGS \times B^mGS & \xrightarrow{\smile^{GS}_{n,m}} & B^{n+m}GS \\
\cong \downarrow & & \\
GB^nS \times GB^mS & & \downarrow \cong \\
\cong \downarrow & & \\
G(B^nS \times B^mS) & \xrightarrow[G(\smile^{S}_{n,m})]{} & G(B^{n+m}S)
\end{array}$$

which holds since the definition of the $\smile$-multiplications only involves arrows which commute with G and B. □

REMARK 4.4. We really need that G be cartesian, since we need it to preserve ring objects, but we could ask that F be merely normal lax symmetric monoidal. This affects the proof only in that the isomorphism in (4.3) becomes just a morphism.

REMARK 4.5. One could start with less than a ring in $\mathcal{C}$ and still be able to carry out the construction above. An interesting example is given by considering rigs, i.e. objects which are like rings but where additive inverses are missing. There is a corresponding functor B^* from rigs in $\mathcal{C}$ to graded rigs in $\mathcal{C}$.

2. Cocommutative comonoids

If our categories are symmetric monoidal but not cartesian, we cannot *a priori* carry out the construction of the previous section. However, note that $s\mathbf{Set}$ *is* cartesian, therefore by Corollary 1.23, for any strong symmetric monoidal functor $F : s\mathbf{Set} \to \mathcal{V}$ we obtain a cartesian functor between cartesian categories

$$F : s\mathbf{Set} \to \mathbf{CoComon}(\mathcal{V}).$$

Moreover, from the dual of Proposition 1.13, if $G : \mathcal{V} \to \mathcal{W}$ is strong symmetric monoidal, we also obtain a cartesian functor

$$G : \mathbf{CoComon}(\mathcal{V}) \to \mathbf{CoComon}(\mathcal{W}).$$

Thus, from Theorem 4.1 we obtain the following

COROLLARY 4.6. *(1) Let $\mathcal{V}$ be a cocomplete closed symmetric monoidal category and $F : s\mathbf{Set} \to \mathcal{V}$ be a strong symmetric monoidal functor which is a left adjoint. There is an iterated bar construction functor*

$$B^* : \mathbf{Ring}(\mathbf{CoComon}(\mathcal{V})) \to \mathbf{GrRing}(\mathbf{CoComon}(\mathcal{V})). \tag{4.7}$$

(2) Further, let $\mathcal{W}$ be a cocomplete closed symmetric monoidal category and $G : \mathcal{V} \to \mathcal{W}$ be a strong symmetric monoidal functor which is a left adjoint. There is a natural isomorphism

$$\begin{array}{ccc} \mathbf{Ring}(\mathbf{CoComon}(\mathcal{V})) & \xrightarrow{B^*} & \mathbf{GrRing}(\mathbf{CoComon}(\mathcal{V})) \\ {\scriptstyle G}\downarrow & \Uparrow & \downarrow{\scriptstyle G} \\ \mathbf{Ring}(\mathbf{CoComon}(\mathcal{W})) & \xrightarrow[B^*]{} & \mathbf{GrRing}(\mathbf{CoComon}(\mathcal{W})). \end{array}$$

Definition 4.8. A graded ring object in cocommutative comonoids in $\mathcal{V}$ is called a *coalgebraic* (or *Hopf*) *ring in* $\mathcal{V}$. If $\mathcal{V}$ is a category of modules over some commutative monoid A, then we call a coalgebraic ring in $\mathcal{V}$ an A-*coalgebraic ring.*

The notion of coalgebraic ring in a symmetric monoidal category of modules over a commutative ring (in the traditional, algebraic sense) was introduced in [**RW77**]: they called it "Hopf ring". We prefer this other term (which they also considered but didn't keep) since it is more explicit.

We can apply this Corollary to a strong symmetric monoidal, left adjoint functor $F : s\mathbf{Set} \to \mathcal{V}$, in which case the situation simplifies: there is a natural isomorphism

$$\begin{array}{ccc} s\mathbf{Ring} & \xrightarrow{B^*} & s\mathbf{GrRing} \\ {\scriptstyle F}\downarrow & \Uparrow & \downarrow{\scriptstyle F} \\ \mathbf{Ring}(\mathbf{CoComon}(\mathcal{V})) & \xrightarrow[B^*]{} & \mathbf{GrRing}(\mathbf{CoComon}(\mathcal{V})). \end{array} \tag{4.9}$$

Remark 4.10. There is a forgetful functor

$$\mathbf{Ring}(\mathbf{CoComon}(\mathcal{V})) \to \mathbf{Ab}(\mathbf{CoComon}(\mathcal{V})) = \mathbf{BiHopf}(\mathcal{V}).$$

Thus, the functor (4.7) has each of its levels B^n forget down to the respective iteration of (3.21).

CHAPTER 5

Examples

We will now give examples of symmetric monoidal categories (Chapter 1), geometric realizations induced by the Yoneda embedding $\Delta^\bullet : \mathbb{\Delta} \to s\mathbf{Set}$ (Chapter 3, Section 2), the reduced bar construction (Chapter 3, Section 4.1) and the graded multiplication on iterated reduced bar constructions (Chapter 4).

1. Simplicial sets

We start with the cartesian closed category $s\mathbf{Set}$ itself. First, some notation: let $s^2\mathcal{C} = \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ be the category of bisimplicial objects in $\mathcal{C}$. By adjunction, we can identify $s^2\mathcal{C}$ with $s(s\mathcal{C})$. Thus, we will think of a bisimplicial object $X_{\bullet,\bullet} \in s^2\mathcal{C}$ as a simplicial object $([n] \mapsto X_{n,\bullet})$ in $s\mathcal{C}$.

Endowed with the Yoneda embedding as a cosimplicial simplicial set, the induced geometric realization

$$|-|_{\Delta^\bullet} : s^2\mathbf{Set} \to s\mathbf{Set}$$

is naturally isomorphic to the diagonal functor: this is well-known (it appears for example as [**GJ99**, Exercise IV.1.4]), but let us now prove it. In symbols,

$$\int^n X_{n,\bullet} \times \Delta^n \cong \mathrm{diag}(X).$$

This is an application of the density theorem (3.8), of the coend formula (3.10) for $\Delta^n \times \Delta^m$, and of Fubini's theorem for coends:

$$\begin{aligned}
\int^n X_{n,\bullet} \times \Delta^n &\cong \int^n \left(\int^m X_{n,m} \cdot \Delta^m \right) \times \Delta^n \\
&\cong \int^{n,m} X_{n,m} \cdot (\Delta^m \times \Delta^n) \\
&\cong \int^{n,m} X_{n,m} \cdot \int^i (\mathbb{\Delta} \times \mathbb{\Delta})((i,i),(n,m)) \cdot \Delta^i \\
&\cong \int^i \left(\int^{n,m} X_{n,m} \times (\mathbb{\Delta} \times \mathbb{\Delta})((i,i),(n,m)) \right) \cdot \Delta^i \\
&\cong \int^i X_{i,i} \cdot \Delta^i \cong \mathrm{diag}(X).
\end{aligned}$$

The bar construction functor (3.18) in this scenario is

$$B : s\mathbf{Ab} \to s\mathbf{Ab}. \tag{5.1}$$

It is a classical result that this functor is weakly homotopy equivalent to the $\bar{W}$-construction of Eilenberg and Mac Lane [**EML53**]. Some discussion and references for this can be found in [**Ste12**] after Lemma 15. One way of obtaining this result is as follows. Duskin identified the functor $\bar{W} : s\mathbf{Ab} \to s\mathbf{Ab}$ with the functor $TB_\bullet : s\mathbf{Ab} \to s\mathbf{Ab}$, where $T : s^2\mathbf{Ab} \to s\mathbf{Ab}$ is a functor going by several names, two of which are "Artin-Mazur diagonal" and "totalization". Then one needs to provide a natural weak homotopy equivalence $T \Rightarrow \mathrm{diag}$. This result has a complicated history: we refer to the aforementioned discussion by Stevenson, and to the more recent [**Zis15**]. If G is an abelian group and we view it as a constant simplicial abelian group, then Eilenberg and Mac Lane proved that $\bar{W}^n G$ gives a simplicial abelian group model for an Eilenberg-Mac Lane space of type $K(G, n)$; see [**GM74**, A.21].

Theorem 4.1 gives us a functor $B^* : \mathbf{Ring}(s\mathbf{Set}) \to \mathbf{GrRing}(s\mathbf{Set})$, i.e.

$$B^* : s\mathbf{Ring} \to s\mathbf{GrRing}. \tag{5.2}$$

When S is a constant simplicial ring, $B^n S$ is a simplicial model for $K(S, n)$, and the graded multiplication is a simplicial model for the cup product in Eilenberg-Mac Lane simplicial sets, as we will see in the next section.

2. Topological spaces

We consider the cartesian closed category **Top** of compactly generated weakly Hausdorff spaces. For an exposition, see [**Str09**].

Let

$$F = |-|^e : s\mathbf{Set} \to \mathbf{Top}$$

be the "extrinsic" geometric realization functor. It is cartesian: this is a well-known result of Milnor [**Mil57**] on the geometric realization of a product of simplicial sets. It is also a left adjoint: its right adjoint is the singular functor.

Note that this extrinsic geometric realization functor is defined following the pattern of Remark 3.5.2: the base (cartesian) monoidal category is **Set**, and we consider **Top** as a **Set**-category.

The cosimplicial object $|\Delta^\bullet|^e$ is the standard cosimplicial space, i.e. $|\Delta^n|^e$ is the topological n-simplex. Therefore the geometric realization $|-|_{|\Delta^\bullet|^e} : s\mathbf{Top} \to \mathbf{Top}$ is the standard geometric realization of a simplicial *space*, as considered e.g. in [**May72**]. It should be noted that our Theorem 3.9 gives a categorical proof that it preserves products (compare with [**May72**, 11.5]): the topology is contained entirely in Milnor's theorem that $|-|^e$ is cartesian.

The resulting functor

$$B : \mathbf{Ab}(\mathbf{Top}) \to \mathbf{Ab}(\mathbf{Top})$$

is Milgram's [**Mil67**] version of the bar construction of a topological abelian group, as observed by Mac Lane [**ML70**]. The space BG is an especially nice model for the classifying space of G, because it carries a strict topological abelian group structure. Thus if $G \in \mathbf{Ab}(\mathbf{Top})$ is discrete, then BG is a model for the Eilenberg-Mac Lane space $K(G,1)$, and B^nG is a model for a $K(G,n)$.

Theorem 4.1 applied to $F = \mathrm{id}_{s\mathbf{Set}}$ and $G = |-|^e : s\mathbf{Set} \to \mathbf{Top}$ gives a natural isomorphism

$$\begin{array}{ccc} s\mathbf{Ring} & \xrightarrow{B^*} & \mathbf{GrRing}(s\mathbf{Set}) \\ {\scriptstyle |-|^e}\downarrow & \Nearrow & \downarrow{\scriptstyle |-|^e} \\ \mathbf{Ring}(\mathbf{Top}) & \xrightarrow[B^*]{} & \mathbf{GrRing}(\mathbf{Top}). \end{array}$$

In other words, we have natural isomorphisms

$$B^n(|S|^e) \cong |B^nS|^e \tag{5.3}$$

compatible with the graded multiplications existing on each side, for $S \in s\mathbf{Ring}$.

When $S \in \mathbf{Ring}(\mathbf{Top})$ is a discrete topological ring, $B^*S = (K(S,n))_{n\geq 0}$ where $K(S,n)$ denotes an n-th Eilenberg-Mac Lane space of S, and the graded multiplication

$$\smile : K(S,n) \times K(S,m) \to K(S,n+m) \tag{5.4}$$

represents the cup product in ordinary cohomology with coefficients in S [**RW80**, 1.7]. Thus, viewing S as a constant simplicial ring, the graded multiplication in simplicial sets

$$B^nS \times B^mS \to B^{n+m}S \tag{5.5}$$

coincides with the cup product map (5.4) after geometric realization under the isomorphism (5.3), i.e. we have gotten a simplicial construction of the cup product map in Eilenberg-Mac Lane simplicial sets.

Let us pass to homology. Let E be a commutative ring spectrum. We denote by $E_* = \pi_*(E)$ its graded commutative ring of coefficients. Let $E_*(-) : \mathbf{Top} \to E_*\text{-}\mathbf{Mod}$ denote its associated unreduced homology theory on spaces taking values in E_*-graded modules.

The category $E_*\text{-}\mathbf{Mod}$ is symmetric monoidal with the tensor product $\otimes_{E_*}$. The functor $E_*(-)$ has a lax symmetric monoidal structure given by the homological cross product

$$E_*(X) \otimes_{E_*} E_*(Y) \to E_*(X \times Y). \tag{5.6}$$

Suppose E satisfies a Künneth isomorphism, i.e. (5.6) is an isomorphism for all spaces X and Y. In other words, $E_*(-)$ is a strong symmetric monoidal functor. As per Corollary 1.23, we get an induced cartesian functor $E_* : \mathbf{Top} \to E_*\text{-}\mathbf{CoCoalg}$, inducing a functor

$$E_* : \mathbf{GrRing}(\mathbf{Top}) \to \mathbf{GrRing}(E_*\text{-}\mathbf{CoCoalg}). \tag{5.7}$$

Thus for a topological ring S, $(E_*(B^n S))_{n\geq 0}$ is a graded E_*-coalgebraic ring (Definition 4.8). For S discrete this was discussed by Ravenel and Wilson [**RW80**].

As a particularly simple case, take $E = Hk$, the Eilenberg-Mac Lane spectrum of the commutative ring k. If k is a field, then Hk satisfies a Künneth isomorphism, but we will not need this in what follows.

Considering sets as discrete topological spaces and modules as graded modules concentrated in degree zero, the functor $(Hk)_* : \mathbf{Top} \to (Hk)_*\text{-}\mathbf{CoCoalg}$ restricts to the functor

$$k[-] : \mathbf{Set} \to k\text{-}\mathbf{CoCoalg} \tag{5.8}$$

which maps a set X to the free k-module $k[X]$ together with the comultiplication obtained by extending linearly the diagonal map on basis elements, $\Delta(x) = x \otimes x$ for $x \in X$. In other words, the cartesian functor (5.8) is obtained from the strong symmetric monoidal free functor $k[-] : \mathbf{Set} \to \mathbf{k\text{-}Mod}$ by passing to cocommutative comonoids.[1]

Thus, if S is a discrete ring, then $(Hk)_*(S)$ is the group ring[2] $k[S]$ as a k-coalgebraic ring (i.e. coalgebraic ring in **k-Mod**) concentrated in degree zero. This is the correct way of characterizing all the structure of the object $k[S]$: in particular, of characterizing the distributivity of the operations coming from the sum and multiplication of S.

It should be noted that there are not that many topological rings, since if G is an abelian topological group then G is homotopy equivalent to $\prod_{n\geq 0} K(\pi_n G, n)$. This result is due to Moore [**Moo55**]; see also [**DT58**, Satz 7.1] or [**Hat02**, Theorem 4.K.6] for a more modern reference.

[1] Remark that (5.8) cannot rightly be called a "free functor", since it is not the left adjoint to the "underlying set" functor: rather, it is the left adjoint to the "set of group-like elements" functor, which maps a coalgebra C to the set of elements $c \in C$ such that $\Delta(c) = c \otimes c$.

[2] More accurately, to be coherent with the naming convention for this kind of object, we should say "the ring k-coalgebraic ring $k[S]$".

3. Simplicial modules and Hochschild homology

Consider the closed symmetric monoidal category of k-modules $\mathcal{V} = \mathbf{k\text{-}Mod}$ with the tensor product $\otimes_k$. Then $\mathbf{CMon}(\mathbf{k\text{-}Mod})^{\mathrm{aug}}$ is the category $k\text{-}\mathbf{CAlg}^{\mathrm{aug}}$ of commutative augmented k-algebras, i.e. commutative k-algebras A with a k-algebra homomorphism $A \to k$. The functor

$$B_\bullet : k\text{-}\mathbf{CAlg}^{\mathrm{aug}} \to s(k\text{-}\mathbf{CAlg}^{\mathrm{aug}}) \tag{5.9}$$

coincides with the reduced simplicial Hochschild functor $HH_\bullet(-,k)$. Indeed, from Corollary 2.25 we know that $B_\bullet(A) \cong B_\bullet^{\mathrm{cy}}(A,k)$, and this is exactly $HH_\bullet(A,k)$, by definition (see e.g. [**Lod98**, Chapter 1]).

However, we cannot go any further with this example: we do not have a natural choice of a functor F as in Chapter 3, Section 2 from simplicial sets to k-modules. To put it differently, we have not been able to find a choice of a cosimplicial module which would yield an interesting realization of a simplicial module into a module.

We thus shift our attention to $\mathcal{V} = s(\mathbf{k\text{-}Mod})$, the closed symmetric monoidal category of simplicial k-modules with pointwise tensor product. It admits a natural strong symmetric monoidal functor from $s\mathbf{Set}$: the free simplicial k-module functor

$$F = k[-] : s\mathbf{Set} \to s(\mathbf{k\text{-}Mod}).$$

It has as right adjoint the functor that forgets the module structure at each level.

Similarly to the case of simplicial sets (Section 1 of the present chapter), the induced geometric realization functor

$$|-|_{k[\Delta^\bullet]} : s^2(\mathbf{k\text{-}Mod}) \to s(\mathbf{k\text{-}Mod})$$

is the diagonal functor. The proof is very similar, but uses instead the *enriched* density theorem [**Kel05**, 3.72].

As remarked in Remark 2.11, the functor $B_\bullet : s(k\text{-}\mathbf{CAlg}^{\mathrm{aug}}) \to s^2(k\text{-}\mathbf{CAlg}^{\mathrm{aug}})$ is, degreewise, the simplicial bar construction (5.9). After geometric realizing (taking diagonals), we obtain a functor

$$B : s(k\text{-}\mathbf{CAlg}^{\mathrm{aug}}) \to s(k\text{-}\mathbf{CAlg}^{\mathrm{aug}}).$$

It is weakly homotopy equivalent to the algebraic $\bar{W}$-construction of Eilenberg and Mac Lane (see [**GM74**], A.14 for the definition of $\bar{W}$ and A.20 for a proof of $BA \simeq \bar{W}A$).

Proposition 3.19 applied to $F = \mathrm{id}_{s\mathbf{Set}}$ and to $G = k[-]$ gives a natural isomorphism

$$\begin{array}{ccc} s\mathbf{CMon}^{\mathrm{aug}} & \xrightarrow{B} & s\mathbf{CMon}^{\mathrm{aug}} \\ {\scriptstyle k[-]}\downarrow & \nearrow\!\!\!\!\nearrow_{\cong} & \downarrow{\scriptstyle k[-]} \\ s(k\text{-}\mathbf{CAlg}^{\mathrm{aug}}) & \xrightarrow[B]{} & s(k\text{-}\mathbf{CAlg}^{\mathrm{aug}}). \end{array}$$

We cannot naively apply Theorem 4.1 to this situation because **k-Mod** is not cartesian. However, we can apply (4.9) to obtain a natural isomorphism

$$\begin{array}{ccc} s\mathbf{Ring} & \xrightarrow{B^*} & s\mathbf{GrRing} \\ {\scriptstyle k[-]}\downarrow & \Uparrow & \downarrow{\scriptstyle k[-]} \\ s\mathbf{Ring}(k\text{-}\mathbf{CoCoalg}) & \xrightarrow[B^*]{} & s\mathbf{GrRing}(k\text{-}\mathbf{CoCoalg}). \end{array}$$

The B^* in the upper line is (5.2). Thus, if S is a simplicial ring, then $k[B^*S] \cong B^*k[S]$ as simplicial graded coalgebraic rings. Let us see how the graded multiplication in $B^*k[S]$ passes to homotopy, just as we passed to homology in Section 2 of the present chapter. For this we need k to be a field, so that the functor

$$\pi_* : s(\mathbf{k\text{-}Mod}) \to \mathbf{Gr}(\mathbf{k\text{-}Mod})$$

is strong symmetric monoidal. Indeed, we can decompose π_* as the normalized Moore functor N into chain complexes (which is lax symmetric monoidal with the shuffle product; see the following section for more details) followed by the homology functor. The Eilenberg-Zilber theorem (see e.g. [**May67**, 29.10]), stating that the shuffle product $NA \otimes NB \to N(A \otimes B)$ is a chain homotopy equivalence, and the algebraic Künneth theorem, giving that the homology functor is strong symmetric monoidal, prove that π_* is strong symmetric monoidal. Thus, we get a functor

$$\pi_* : s\mathbf{GrRing}(k\text{-}\mathbf{CoCoalg}) \to \mathbf{GrRing}(k\text{-}\mathbf{CoCoalg}).$$

Let $A \in s\mathbf{Ring}(k\text{-}\mathbf{CoCoalg})$ be a constant simplicial object. By neglect of structure, A is an augmented commutative k-algebra, where the augmentation is given by the counit. Then the homotopy of B^nA gives $HH_*^{[n]}(A,k)$, Pirashvili's [**Pir00**] higher order reduced Hochschild homology of A, as noted in [**LR11**, Section 3.1]. The multiplication of A induces a graded multiplication

$$HH_*^{[n]}(A,k) \otimes HH_*^{[m]}(A,k) \to HH_*^{[n+m]}(A,k). \tag{5.10}$$

Now let S be a constant simplicial ring and consider $A = k[S]$. As seen in Section 2 of the present chapter, B^*S is a model for the Eilenberg-Mac Lane graded simplicial ring $K(S,*)$. Thus, we have an isomorphism

$$HH_*^{[*]}(k[S],k) \cong k[K(S,*)]$$

and the graded multiplication (5.10) corresponds to the graded multiplication (5.5) corresponding to the cup product in cohomology.

The reader might want to jump to Section 5.3 of the present chapter where we analyze the analogous phenomena happening in *topological* Hochschild homology.

4. Differential graded modules

Let $\mathcal{W} = \mathbf{k\text{-}dgm}$ be the closed symmetric monoidal category of non-negatively graded differential graded k-modules (chain complexes) with the tensor product $\otimes_k$. We would like the normalized Moore functor $N : s(\mathbf{k\text{-}Mod}) \to \mathbf{k\text{-}dgm}$ (see below) to play the role of the functor called G in Chapter 3, Section 3; however, it is not *strong* symmetric monoidal, it is merely colax (with an Alexander-Whitney map) and symmetric lax (with a shuffle product map) which is why we cannot apply the machinery of Chapter 4, most notably Corollary 4.6. However, since this is a very classically useful set of tools whose explicit details are often not written down, we choose to expand them.

First, define a functor $M : s(\mathbf{k\text{-}Mod}) \to \mathbf{k\text{-}dgm}$, the *unnormalized Moore functor*, by $(MX)_p = X_p$ for all $p \geq 0$ with differential $\sum_{i=0}^{n}(-1)^i d_i$ in degree n. Now, define the *normalized Moore functor*

$$N : s(\mathbf{k\text{-}Mod}) \to \mathbf{k\text{-}dgm}$$

by $(NX)_0 = X_0$ and $(NX)_n = \bigcap_{i=0}^{n-1} \ker(d_i : X_n \to X_{n-1})$ for all $n \geq 1$. Since the n-th differential of $(MX)_n$ sends $(NX)_n$ into $(NX)_{n+1}$, we get an induced differential on the graded module NX.

The functor M is lax symmetric monoidal, with the *shuffle map* (also called *Eilenberg-Zilber map*) $MX \otimes MY \to M(X \otimes Y)$ defined by

$$X_p \otimes Y_q \to X_{p+q} \otimes Y_{p+q}$$

$$x \otimes y \mapsto \sum_{(\mu,\nu)\in \mathrm{Sh}_{p+q}} (-1)^{\mathrm{sg}(\mu,\nu)} s_{\nu_q-1} \dots s_{\nu_1-1}(x) \otimes s_{\mu_p-1} \dots s_{\mu_1-1}(y).$$

Here Sh_{p+q} denotes the set of (p,q)-shuffles and $\mathrm{sg}(\mu,\nu)$ denotes the sign of the $(p+q)$-permutation associated to the (p,q)-shuffle (μ,ν). The unit map $\underline{k} \to M(ck)$ of M, where $\underline{k} \in \mathbf{k\text{-}dgm}$ is k concentrated in degree zero and ck is the constant simplicial object at k, is the identity in degree zero and zero otherwise.

The functor M is also colax monoidal (but not symmetric), with the *Alexander-Whitney map* $M(X \otimes Y) \to MX \otimes MY$ given by

$$X_n \otimes Y_n \to \bigoplus_{i=0}^{n} X_i \otimes Y_{n-i}$$

$$x \otimes y \mapsto \sum_{i=0}^{n} d_{i+1} \dots d_n(x) \otimes d_0 \dots d_{i-1}(y).$$

The counit $M(ck) \to \underline{k}$ is the identity in degree zero and zero otherwise.

A detailed proof that M together with these maps really is lax symmetric and colax can be found in [**AM10**, Theorem 5.6]: the authors prove that M is even *bilax*, a condition which we have not introduced and for which we will have no need, which guarantees the compatibility of these two structures.

It can be proven that, for a simplicial module X, we have $X_n = (NX)_n \oplus (DX)_n$, where $(DX)_n = \sum_{i=0}^{n-1} s_i(X_{n-1})$ [**Wei94**, Lemma 8.3.7]. Under the projections $X_n \to (NX)_n$, the functor N gets induced lax symmetric and colax strutures from those of M, and moreover both of these structures are normal [**AM10**, Corollary 5.7].

The functor N can be seen as an extrinsic geometric realization, following Remark 3.5.2. It can be considered a sort of linearization of $|-|^e : s\mathbf{Set} \to \mathbf{Top}$. Indeed, the category of chain complexes is enriched, tensored and cotensored over modules. The cosimplicial chain complex $Nk[\Delta^\bullet] : \mathbb{\Delta} \to \mathbf{k\text{-}dgm}$ begets a functor

$$|-|^e_{Nk[\Delta^\bullet]} : s(\mathbf{k\text{-}Mod}) \to \mathbf{k\text{-}dgm}.$$

The fact that $N \cong |-|^e_{Nk[\Delta^\bullet]}$ was observed in [**Kan58**, 6.3] by direct computation. Alternatively, if we know that the Dold-Kan correspondence is an *adjoint* equivalence [**Wei94**, 8.4.2], then just as in Proposition 3.3 we get that the right adjoint to $|-|^e_{Nk[\Delta^\bullet]}$ is the functor $A \mapsto \underline{\mathbf{k\text{-}dgm}}(Nk[\Delta^\bullet], A)$. This functor is known to be the right adjoint to N in the Dold-Kan correspondence, therefore $N \cong |-|^e_{Nk[\Delta^\bullet]}$ by uniqueness of left adjoints.

The identification $N \cong |-|^e_{Nk[\Delta^\bullet]}$ is a priori a bit unsatisfactory, since to define $|-|^e_{Nk[\Delta^\bullet]}$ we use the cosimplicial chain complex $Nk[\Delta^\bullet]$ which depends on N. However, $Nk[\Delta^\bullet]$ can be defined independently: for example, it is the cellular chain complex on the cosimplcial space $|\Delta^\bullet|^e$.

At any rate, the cosimplicial chain complex $Nk[\Delta^\bullet] : \mathbb{\Delta} \to \mathbf{k\text{-}dgm}$ yields an interesting geometric realization

$$|-|_{Nk[\Delta^\bullet]} : s(\mathbf{k\text{-}dgm}) \to \mathbf{k\text{-}dgm}.$$

Indeed, $|-|_{Nk[\Delta^\bullet]}$ coincides with the functor $C : s(\mathbf{k\text{-}dgm}) \to \mathbf{k\text{-}dgm}$ which is the composition of the functors "normalized Moore in each internal degree", yielding a bicomplex, and the "totalization of a bicomplex" functor. One can prove that $C \cong |-|_{Nk[\Delta^\bullet]}$ by a computation similar to the one by Kan which proves that $N \cong |-|^e_{Nk[\Delta^\bullet]}$.

Gugenheim and May [**GM74**, A.2] call the functor C *condensation*, and they prove directly in Proposition A.3 that it is a normal lax symmetric monoidal functor, via a suitable totalized shuffle product, and colax, via a suitable Alexander-Whitney map.

Note that the induced bar construction functor on augmented commutative differential graded k-algebras

$$B : k\text{-}\mathbf{cdga}^{\mathrm{aug}} \to k\text{-}\mathbf{cdga}^{\mathrm{aug}} \tag{5.11}$$

is the classical bar construction of such objects, as introduced in [**EML53**, Theorem 11.1] and considered e.g. in [**GM74**, Page 69]. The multiplication of BA is the "shuffle product", since it is induced by the lax structure of N also called thusly. We can be much more explicit, and for the sake of examples, let us be so. We draw this explicit description from [**EML53**].

Let $A \in k\text{-}\mathbf{dga}^{\mathrm{aug}}$ with augmentation $\varepsilon : A \to k$. First, consider $\hat{B}A := \bigoplus_{n\geq 0} A^{\otimes n}$. Following the original notation of Eilenberg and Mac Lane which is at the origin of the name of the construction, denote an elementary tensor in $\hat{B}A$ by $[a_1|\cdots|a_n]$. We give $\hat{B}A$ the graduation $|-|_B = |-|_i + |-|_s$: here $|[a_1|\cdots|a_n]|_i = \sum_{i=1}^n |a_i|$ is the *internal graduation*, and $|[a_1|\cdots|a_n]|_s = n$ is the *simplicial graduation*. Thus,

$$\hat{B}A_d = \bigoplus_{m+n=d} \bigoplus_{i_1+\cdots+i_m=n} (A_{i_1} \otimes \cdots \otimes A_{i_m}).$$

We give $\hat{B}A$ the differential $d_B = d_i + d_s$, where

$$d_i([a_1|\cdots|a_n]) = -\sum_{i=1}^{n} (-1)^{e_{i-1}} [a_1|\cdots|d(a_i)|\cdots|a_n],$$

$$\begin{aligned} d_s([a_1|\cdots|a_n]) &= \varepsilon(a_1)[a_2|\cdots|a_n] \\ &+ \sum_{i=1}^{n-1} (-1)^{e_i} [a_1|\cdots|a_i a_{i+1}|\cdots|a_n] \\ &+ (-1)^{e_n} [a_1|\cdots|a_{n-1}]\varepsilon(a_n), \end{aligned}$$

and $e_i = |[a_1|\cdots|a_i]|_B$. The augmentation of $\hat{B}A$ is $\varepsilon_B : \hat{B}A \to \underline{k}$, $\lambda \mapsto \lambda$ if $\lambda \in k$ and $\lambda \mapsto 0$ otherwise.

If A is commutative, then $\hat{B}A$ gets the *shuffle product* given by

$$[a_1|\cdots|a_m][b_1|\cdots|b_n] = \sum_{\pi \in \mathrm{Sh}_{m,n}} (-1)^{\mathrm{sg}(\pi,a,b)} [c_{\pi(1)}|\cdots|c_{\pi(m+n)}]$$

where $(c_1, \ldots, c_{m+n}) = (a_1, \ldots, a_m, b_1, \ldots, b_n)$ and $\mathrm{sg}(\pi, a, b) = \sum_{\substack{i,j \\ \pi(i)>\pi(m+j)}} |a_i|_B |b_j|_B$. This finishes the concrete description of the functor $\hat{B} : k\text{-}\mathbf{cdga}^{\mathrm{aug}} \to k\text{-}\mathbf{cdga}^{\mathrm{aug}}$. To recover the functor (5.11), set $BA = \hat{B}A/s\hat{B}A$, the quotient of BA by the subcomplex generated by the elements of the form $[a_1|\cdots|1_A|\cdots|a_n]$.

Finally, note that $\hat{B}A$ is nothing other than $\hat{C}B_\bullet A$, where $\hat{C}$ is the "unnormalized condensation functor", i.e. unnormalized Moore in each internal degree, then totalization of the bicomplex. This is the geometric realization functor obtained from using $Mk[\Delta^\bullet]$ as a cosimplicial chain complex.

In this differential graded setting, the bar construction has more structure. If $A \in k\text{-}\mathbf{dga}^{\mathrm{aug}}$, then $\hat{B}A$ has a non-cocommutative comultiplication Δ_B [**GM74**, Page 72] sometimes called *deconcatenation*, given by

$$\Delta_B([a_1|\cdots|a_n]) = \sum_{i=0}^{n} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n]$$

such that $\hat{B}A$ together with this comultiplication and the augmentation described above as counit is a differential graded coalgebra. If A is commutative, then $\hat{B}A$ is a differential graded commutative Hopf algebra. By passing to the quotient, the analogous statements hold for BA instead of $\hat{B}A$.

Now, suppose A is a differential graded commutative k-bialgebra. This can be phrased as: A is a comonoid in $k\text{-}\mathbf{cdga}^{\mathrm{aug}}$. The bar construction functor (5.11) is colax monoidal, since $B_\bullet$ is strong monoidal and $|-| = C$ is colax monoidal, therefore BA has an induced comultiplication. We will now prove that it coincides with Δ_B, or rather, we will do this for $\hat{B}A$, for simplicity. Let Δ denote the comultiplication of A. For $a_i \in A$, write $\Delta a_i = \sum a_i' \otimes a_i''$. Then the induced comultiplication on $\hat{B}A$ is

$$\hat{B}A \xrightarrow{\hat{B}\Delta} \hat{B}(A \otimes A) \xrightarrow{AW} \hat{B}A \otimes \hat{B}A$$

$$[a_1|\cdots|a_n] \mapsto \sum [a_1' \otimes a_1''|\cdots|a_n' \otimes a_n''] \mapsto$$

$$\sum_{i=0}^{n} \sum [\varepsilon(a_1'')a_1'|\cdots|\varepsilon(a_i'')a_i'] \otimes [\varepsilon(a_i')a_{i+1}''|\cdots \varepsilon(a_n')a_n''].$$

Now note that the counitality conditions for Δ and ε give that $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$, i.e. $a_i = \sum \varepsilon(a_i'')a_i'$. Therefore, this induced comultiplication on $\hat{B}A$ coincides with Δ_B. This argument works just as well for BA instead of $\hat{B}A$. We thank Benoît Fresse for pointing out the previous argument to us.

As a final remark, recall Dold and Puppe's version of the Eilenberg-Zilber theorem [**GJ99**, IV.2.4], which states that there is a natural quasi-isomorphism

$$N(\operatorname{diag} X_{\bullet,\bullet}) \to CN(X_{\bullet,\bullet})$$

for a bisimplicial module $X_{\bullet,\bullet}$. Rephrasing it using geometric realizations, there is a natural quasi-isomorphism $N|-| \Rightarrow |N-|$. It would be interesting to check whether it is a *monoidal* isomorphism.

5. Brave new algebra and topological Hochschild homology

5.1. Spectra. Let $\mathcal{W}$ be the closed symmetric monoidal category $(\mathbb{S}\text{-}\mathbf{Mod}, \wedge_{\mathbb{S}}, \mathbb{S})$ of S-modules of [**EKMM97**]. In this section, a *spectrum* will mean an S-module, and a *(commutative) ring spectrum* will mean a (commutative) S-algebra.

We take as a functor G in the sense of Chapter 3, Section 3 the functor

$$\Sigma^{\infty}_{+} : \mathbf{Top} \to \mathbb{S}\text{-}\mathbf{Mod} \tag{5.12}$$

which maps a topological space X to the suspension spectrum on the space X with an added disjoint basepoint. It is strong symmetric monoidal [**EKMM97**, II.1.2], and it is a left adjoint [**EKMM97**, Page 39]. We therefore consider $\mathbb{S}$-**Mod** endowed with the cosimplicial spectrum $\Sigma^{\infty}_{+}|\Delta^{\bullet}|^{e} : \mathbb{\Delta} \to \mathbb{S}\text{-}\mathbf{Mod}$.

It should be noted that if $X \in \mathbf{Top}$ and $C \in \mathbb{S}\text{-}\mathbf{Mod}$, then $C \wedge \Sigma^{\infty}_{+}X$ defines the standard tensoring of $\mathbb{S}$-**Mod** over **Top** [**EKMM97**, III.1.1]. Therefore, the induced geometric realization

$$|-|_{\Sigma^{\infty}_{+}|\Delta^{\bullet}|^{e}} =: |-| : s(\mathbb{S}\text{-}\mathbf{Mod}) \to \mathbb{S}\text{-}\mathbf{Mod} \tag{5.13}$$

in our sense coincides with the one in [**EKMM97**, X.1.1].

By Theorem 3.9, $|-|$ is strong symmetric monoidal. This appears in [**EKMM97**, X.1.4]. Theorem 3.11 applies to prove that the natural isomorphism $\Sigma^{\infty}_{+}|X_{\bullet}| \cong |\Sigma^{\infty}_{+}X_{\bullet}|$, which appears in [**EKMM97**, X.1.3.i], is monoidal. This has not, to our knowledge, explicitly appeared in the literature.

We could apply Corollary 4.6 to the strong symmetric monoidal functor G of (5.12) right now. But instead, let us first delve into general R-modules and apply the machinery there (see (5.15)): it will give a more general result.

5.2. R-modules and extension of scalars. Let R be a commutative ring spectrum and take $\mathcal{W}$ to be the closed symmetric monoidal category R-**Mod** of R-modules, with smash product $\wedge = \wedge_R$ as tensor product, and as unit the R-module $R \wedge_{\mathbb{S}} \mathbb{S}$.

We take as a functor G in the sense of Chapter 3, Section 3 the strong symmetric monoidal functor of extension of scalars

$$R \wedge_{\mathbb{S}} - : \mathbb{S}\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod},$$

whose right adjoint is the restriction of scalars functor. Thus R-**Mod** is endowed with the cosimplicial R-module $R \wedge_{\mathbb{S}} \Sigma^{\infty}_{+}|\Delta^{\bullet}|^{e}$ (recall from Section 2 of the current chapter that $|-|^{e} : s\mathbf{Set} \to \mathbf{Top}$ is the standard "extrinsic" geometric realization functor).

Furthermore, if R' is another commutative ring spectrum and $f : R \to R'$ is a morphism, then in just the same fashion we obtain a strong symmetric monoidal functor G

$$R' \wedge_R - : R\text{-}\mathbf{Mod} \to R'\text{-}\mathbf{Mod}$$

endowing R'-**Mod** with the cosimplicial R'-module $R' \wedge_R R \wedge_{\mathbb{S}} \Sigma^\infty_+ |\Delta^\bullet|^e \cong R' \wedge_{\mathbb{S}} \Sigma^\infty_+ |\Delta^\bullet|^e$.

Corollary 4.6 gives a natural isomorphism comparing the iterated bar construction together with its graded structure, whenever carried out in R-**Mod** or R'-**Mod**:

$$\begin{array}{ccc} \mathbf{Ring}(R\text{-}\mathbf{CoCoalg}) & \xrightarrow{B^*} & \mathbf{GrRing}(R\text{-}\mathbf{CoCoalg}) \\ {\scriptstyle R'\wedge_R -}\downarrow & \Uparrow & \downarrow{\scriptstyle R'\wedge_R -} \\ \mathbf{Ring}(R'\text{-}\mathbf{CoCoalg}) & \xrightarrow[B^*]{} & \mathbf{GrRing}(R'\text{-}\mathbf{CoCoalg}). \end{array}$$

We will now analyze what the bar construction in this context actually is.

5.3. Topological Hochschild homology. The simplicial bar construction in the category R-**Mod**,

$$B_\bullet : R\text{-}\mathbf{CAlg}^{\mathrm{aug}} \to s(R\text{-}\mathbf{CAlg}^{\mathrm{aug}}),$$

coincides with $THH^R_\bullet(-,R)$, the reduced simplicial topological Hochschild homology functor. Indeed, from Corollary 2.25 we know that $B_\bullet(A) \cong B^{\mathrm{cy}}_\bullet(A,R)$, and this is exactly $THH_\bullet(A,R)$, by definition. Therefore,

$$B : R\text{-}\mathbf{CAlg}^{\mathrm{aug}} \to R\text{-}\mathbf{CAlg}^{\mathrm{aug}}$$

is an explicit model for the reduced topological Hochschild homology functor $THH^R(-,R)$, as noted in [**EKMM97**, IX.2].

Note that if one wants this $THH^R(A,R)$ to coincide with the derived smash product $R \wedge^L_{A^e} A$ using the model structure in R-**Mod** set in [**EKMM97**], it is necessary that R be a q-cofibrant commutative S-algebra and that A be a q-cofibrant commutative R-algebra. This is enough by a slight modification of Theorem 2.6 in [**EKMM97**].

The iterations of B,

$$B^n : R\text{-}\mathbf{CAlg}^{\mathrm{aug}} \to R\text{-}\mathbf{CAlg}^{\mathrm{aug}} \tag{5.14}$$

for $n \geq 0$ are an explicit model for higher reduced topological Hochschild homology $THH^{R,[n]}(-,R)$ as considered e.g. in [**BLP$^+$15**].

Indeed, $THH^R(A,R)$ can be expressed as $S^1 \odot A$, where $\odot$ denotes the tensoring of the category R-$\mathbf{CAlg}^{\mathrm{aug}}$ over pointed topological spaces [**Kuh04**, 7.1], and its higher version is, by definition,

$$THH^{R,[n]}(A,R) = S^n \odot A.$$

We obtain natural isomorphisms

$$\begin{aligned} B^2(A) = THH^R(THH^R(A,R),R) = S^1 \odot (S^1 \odot A) \cong \\ \cong (S^1 \wedge S^1) \odot A \cong S^2 \odot A = THH^{R,[2]}(A,R) \end{aligned}$$

and similarly for higher powers.

We can now apply Corollary 4.6 to the functor

$$G = R[-] = R \wedge_{\mathbb{S}} \Sigma^\infty_+ : \mathbf{Top} \to R\text{-}\mathbf{Mod}, \tag{5.15}$$

and obtain a natural isomorphism

$$\begin{array}{ccc} \mathbf{Ring}(\mathbf{Top}) & \xrightarrow{B^*} & \mathbf{GrRing}(\mathbf{Top}) \\ {\scriptstyle R[-]}\downarrow & \Uparrow & \downarrow{\scriptstyle R[-]} \\ \mathbf{Ring}(R\text{-}\mathbf{CoCoalg}) & \xrightarrow[B^*]{} & \mathbf{GrRing}(R\text{-}\mathbf{CoCoalg}). \end{array}$$

We thus get, for $A \in \mathbf{Ring}(R\text{-}\mathbf{CoCoalg})$, a graded multiplication in higher THH:

$$THH^{R,[n]}(A,R) \wedge_R THH^{R,[m]}(A,R) \to THH^{R,[n+m]}(A,R) \tag{5.16}$$

and if $A = R[S]$ for $S \in \mathbf{Ring}(\mathbf{Top})$, then we get a natural isomorphism

$$THH^{R,[*]}(R[S],R) \cong R[B^*S] \tag{5.17}$$

of graded ring objects of R-**CoCoalg**, i.e. of coalgebraic rings in R-**Mod** (Definition 4.8). As noted in Section 2 of the present chapter, when S is discrete the graded topological ring B^*S is a model for the Eilenberg-Mac Lane spaces $K(S,*)$.

Observe that by means of Remark 4.10, we get for every $n \geq 0$ an isomorphism

$$THH^{R,[n]}(R[G],R) \cong R[B^nG]$$

of bicommutative R-Hopf algebras, natural in the topological abelian group G.

Let us pass to homology. Let $E \in R$-**CAlg** be a field, meaning that $E_* = \pi_* E$ is a graded field, i.e. every graded module over it is free. Then the E-homology functor $E_* : R\text{-}\mathbf{Mod} \to E_*\text{-}\mathbf{Mod}$ is strong symmetric monoidal, since the field hypothesis guarantees a Künneth isomorphism. It induces a functor

$$E_* : \mathbf{GrRing}(R\text{-}\mathbf{CoCoalg}) \to \mathbf{GrRing}(E_*\text{-}\mathbf{CoCoalg}). \tag{5.18}$$

Thus, $E_*(THH^{R,[*]}(R[S],R))$ is an E_*-coalgebraic ring. If in particular we take R to be $\mathbb{S}$, we can recover the coalgebraic ring of Ravenel-Wilson (see Section 2 of the present chapter) in a different guise. Indeed, precomposing (5.18) with

$$\mathbb{S}[-] : \mathbf{GrRing}(\mathbf{Top}) \to \mathbf{GrRing}(R\text{-}\mathbf{CoCoalg})$$

gives the E-homology of topological rings (5.7). Then, if $S \in \mathbf{Ring}(\mathbf{Top})$, we get an isomorphism of E_*-coalgebraic rings

$$E_*(THH^{\mathbb{S},[*]}(\mathbb{S}[S],\mathbb{S})) \cong E_*(B^*S). \tag{5.19}$$

CHAPTER 6

Iterated and higher topological Hochschild homology of KU

In this chapter, we give different expressions for $THH(KU)$ as a commutative KU-algebra. Then we describe the commutative KU-algebras $THH^n(KU)$ and $X \otimes KU$ for based CW-complexes X which are reduced suspensions.

Let us first state a couple of conventions and recall some facts.

By *space* we will mean "compactly generated weakly Hausdorff space", and we will denote the cartesian closed category they form by **Top**. We will work in the categories of [**EKMM97**]: our main objects are $\mathbb{S}$-modules, commutative $\mathbb{S}$-algebras R, R-modules and commutative R-algebras A.

1. Model structures

The category $\mathbb{S}$-**Mod** of $\mathbb{S}$-modules has a topological (i.e. enriched, tensored and cotensored over **Top**) symmetric monoidal cofibrantly generated model structure [**EKMM97**, VII.4]. A commutative $\mathbb{S}$-algebra is, by definition, a commutative monoid in $\mathbb{S}$-**Mod**. The category they form, $\mathbb{S}$-**CAlg**, can also be described as the category of $\mathbb{P}$-algebras where $\mathbb{P}$ is the commutative monoid monad. The forgetful functor $U : \mathbb{S}\text{-}\mathbf{CAlg} \to \mathbb{S}\text{-}\mathbf{Mod}$ creates a model structure on $\mathbb{S}$-**CAlg**[1]. In particular, there is a Quillen adjunction $\mathbb{S}\text{-}\mathbf{Mod} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{S}\text{-}\mathbf{CAlg}$. The category $\mathbb{S}$-**CAlg** has a topological symmetric monoidal cofibrantly generated model category strucutre.

Let $R \in \mathbb{S}\text{-}\mathbf{CAlg}$, and consider the category of R-modules, R-**Mod**. The forgetful functor $R\text{-}\mathbf{Mod} \to \mathbb{S}\text{-}\mathbf{Mod}$ creates a model structure on R-**Mod**, and R-**Mod** acquires a topological symmetric monoidal cofibrantly generated model category structure. The forgetful functor $U : R\text{-}\mathbf{CAlg} \to R\text{-}\mathbf{Mod}$ creates a model structure on R-**CAlg**, and thus R-**CAlg** has a topological symmetric monoidal cofibrantly generated model category structure. In these model categories, all objects are fibrant.

---

[1]A functor $U : \mathcal{C} \to \mathcal{M}$ *creates a model structure* on $\mathcal{C}$ if $\mathcal{M}$ is a model category and $\mathcal{C}$ is a model category such that f is a fibration (resp. weak equivalence) in $\mathcal{C}$ if and only if Uf is a fibration (resp. weak equivalence) in $\mathcal{M}$. We say that U *strongly creates* the model structure of $\mathcal{C}$ if, in addition, f is a cofibration in $\mathcal{C}$ if and only if Uf is a cofibration in $\mathcal{M}$.

Cofibrancy is more delicate. The sphere $\mathbb{S}$-module $\mathbb{S}$ is not cofibrant as an $\mathbb{S}$-module, but it is cofibrant as a commutative $\mathbb{S}$-algebra. More generally, the underlying R-module of a cofibrant commutative R-algebra is generally not cofibrant as an R-module.

Let R be a commutative $\mathbb{S}$-algebra. We record the following useful properties:

(1) If M is a cofibrant R-module, then $M \wedge_R -$ preserves all weak equivalences of R-modules [**EKMM97**, III.3.8].
(2) Suppose R is cofibrant. Let A and B be cofibrant commutative R-algebras. Let $\gamma_A : \Gamma A \to A$ and $\gamma_B : \Gamma B \to B$ be cofibrant approximations of A and B in the category of R-modules. Then $\gamma_A \wedge_R \gamma_B : \Gamma A \wedge_R \Gamma B \to A \wedge_R B$ is a weak equivalence of R-modules [**EKMM97**, VII.6.5, VII.6.7].
(3) As in any model category, the coproduct of cofibrant objects is cofibrant. Hence, if A and B are cofibrant commutative R-algebras, then $A \wedge_R B$ is a cofibrant commutative R-algebra [**EKMM97**, VII.6.8].
(4) Let $A \to B$ be a cofibration of cofibrant commutative R-algebras, where R is a cofibrant comutative $\mathbb{S}$-algebra. Then $B \wedge_A - : A\text{-}\mathbf{CAlg} \to B\text{-}\mathbf{CAlg}$ preserves weak equivalences between commutative A-algebras which are cofibrant as commutative $\mathbb{S}$-algebras [**EKMM97**, VII.7.4].
(5) The category R-**CAlg** can also be described as the category of objects of $\mathbb{S}$-**CAlg** under R. The forgetful functor $R\text{-}\mathbf{CAlg} \to \mathbb{S}\text{-}\mathbf{CAlg}$ thus strongly creates a model structure on R-**CAlg** [**MP12**, Theorem 15.3.6]. This model structure coincides with the one described above [**Hön17**, Remark 2.4.1]. In conclusion, a map $f : A \to B$ is a cofibration in R-**CAlg** if and only if it is a cofibration in $\mathbb{S}$-**CAlg**. In particular, if R is a cofibrant commutative $\mathbb{S}$-algebra and A is a cofibrant commutative R-algebra, then A is cofibrant as a commutative $\mathbb{S}$-algebra.

Note: in [**EKMM97**] they call *q-cofibration* what we call a cofibration. We will have no use for what they call a "cofibration".

2. Inversion of an element

In this section, we describe the procedure of inverting a homotopy element in a commutative $\mathbb{S}$-algebra and prove some properties which will be needed below.

THEOREM 6.1. [**EKMM97**, VIII.2.2, VIII.4.2] *Let R be a cofibrant commutative $\mathbb{S}$-algebra and $x \in \pi_*R$. There exists a cofibrant commutative R-algebra $R[x^{-1}]$ with unit $j : R \to R[x^{-1}]$ satisfying that $\pi_*(R[x^{-1}]) = \pi_*(R)[x^{-1}]$, and if $f : R \to T$ is a map in $\mathbb{S}$-**CAlg** such that $(\pi_*f)(x) \in \pi_*T$ is invertible, then there exists a map $\tilde{f} : R[x^{-1}] \to T$*

in $\mathbb{S}$-**CAlg** *making the following diagram commute.*

$$\begin{array}{ccc} R & \xrightarrow{f} & T \\ {\scriptstyle j}\downarrow & \nearrow_{\tilde{f}} & \\ R[x^{-1}] & & \end{array}$$

Moreover, if the map $\pi_*(R)[x^{-1}] \to \pi_*T$ *coming from the universal property for localizations of commutative* $\pi_*(R)$*-algebras is an isomorphism, then* $\tilde{f}$ *is a weak equivalence.*

The previous theorem is valid, *mutatis mutandis*, if $\mathbb{S}$ is replaced by some cofibrant commutative $\mathbb{S}$-algebra.

LEMMA 6.2. *The multiplication map* $\mu : R[x^{-1}] \wedge_R R[x^{-1}] \to R[x^{-1}]$ *is a weak equivalence of commutative* $R[x^{-1}]$*-algebras.*

PROOF. The Tor spectral sequence [**EKMM97**, IV.4.1] here takes the form

$$E^2_{*,*} = \mathrm{Tor}^{\pi_*R}_{*,*}(\pi_*R[x^{-1}], \pi_*R[x^{-1}]) \Rightarrow \pi_*(R[x^{-1}] \wedge_R R[x^{-1}]).$$

Since the localization morphism $\pi_*R \to \pi_*R[x^{-1}]$ is flat, the spectral sequence is concentrated in the 0-th column and thus the edge homomorphism

$$\nabla : \pi_*R[x^{-1}] \otimes_{\pi_*R} \pi_*R[x^{-1}] \to \pi_*(R[x^{-1}] \wedge_R R[x^{-1}]) \tag{6.3}$$

is an isomorphism. Since $\wedge_R$ is the coproduct in the category of commutative R-algebras, we can consider the canonical maps $i_1, i_2 : R[x^{-1}] \to R[x^{-1}] \wedge_R R[x^{-1}]$. The edge homomorphism ∇ coincides with the map (π_*i_1, π_*i_2) defined via the universal property of the coproduct of commutative π_*R-algebras. We have the following commutative diagram of commutative π_*R-algebras:

$$\begin{array}{ccccc} \pi_*R[x^{-1}] & \xrightarrow{\iota_1} & \pi_*R[x^{-1}] \otimes_{\pi_*R} \pi_*R[x^{-1}] & \xleftarrow{\iota_2} & \pi_*R[x^{-1}] \\ & \searrow^{\pi_*i_1} & \downarrow \nabla & \swarrow^{\pi_*i_2} & \\ & & \pi_*(R[x^{-1}] \wedge_R R[x^{-1}]) & & \\ & {\scriptstyle \mathrm{id}} & \downarrow \pi_*\mu & {\scriptstyle \mathrm{id}} & \\ & & \pi_*R[x^{-1}] & & \end{array}$$

where ι_1, ι_2 are the canonical inclusions into a coproduct of commutative π_*R-algebras. Again, by the universal property of the coproduct of commutative π_*R-algebras, there is a unique arrow $\pi_*R[x^{-1}] \otimes_{\pi_*R} \pi_*R[x^{-1}] \to \pi_*R[x^{-1}]$ making the outer diagram commute. One such arrow is the canonical isomorphism that one has for any such algebraic localization, i.e. $h : S^{-1}A \otimes_A S^{-1}A \overset{\cong}{\to} S^{-1}A$ for any commutative ring A and multiplicative subset $S \subset A$. Another such arrow is $\pi_*\mu \circ \nabla$. Therefore, $h = \pi_*\mu \circ \nabla$. Since ∇ and h are isomorphisms, so is $\pi_*\mu$. □

If $f : R \to T$ is a morphism between cofibrant commutative $\mathbb{S}$-algebras and $x \in \pi_* R$, then Theorem 6.1 gives us a map of cofibrant commutative $\mathbb{S}$-algebras

$$\begin{array}{ccc} R & \xrightarrow{f} & T \\ {\scriptstyle j_R}\downarrow & & \downarrow{\scriptstyle j_T} \\ R[x^{-1}] & \underset{f[x^{-1}]}{\dashrightarrow} & T[(\pi_* f)(x)^{-1}]. \end{array}$$

Note that $f[x^{-1}]$ turns $T[(\pi_* f)(x)^{-1}]$ into a commutative $R[x^{-1}]$-algebra.

The previous square induces an arrow from the pushout $R[x^{-1}] \wedge_R T$ in R-**CAlg**. The following theorem tells us that it is a weak equivalence. Compare with [**EKMM97**, V.1.15] which handles the case where T is replaced by an R-module.

PROPOSITION 6.4 (Base change for localization). *Let $f : R \to T$ be a morphism of cofibrant commutative $\mathbb{S}$-algebras and $x \in \pi_* R$. The morphism of commutative R-algebras*

$$(f[x^{-1}], j_T) : R[x^{-1}] \wedge_R T \to T[(\pi_* f)(x)^{-1}] \tag{6.5}$$

is a weak equivalence.

Note that (6.5) is also an equivalence in $R[x^{-1}]$-**CAlg** and in T-**CAlg**.

PROOF. Denote the morphism $(f[x^{-1}], j_T)$ by h, for simplicity. Like in the proof of Lemma 6.2, the Tor spectral sequence that computes the homotopy groups of $R[x^{-1}] \wedge_R T$ from those of $R[x^{-1}]$ and T collapses, since $\pi_* R \to \pi_* R[x^{-1}] = (\pi_* R)[x^{-1}]$ is flat. Therefore, the map $\pi_* h$, fitting in a commutative diagram

$$\begin{array}{ccc} \pi_*(R[x^{-1}] \wedge_R T) & \xrightarrow{\pi_* h} & (\pi_* T)[(\pi_* f)(x)^{-1}] \\ \cong\uparrow & \nearrow\cong & \\ (\pi_* R)[x^{-1}] \otimes_{\pi_* R} \pi_* T, & & \end{array}$$

is an isomorphism, since the diagonal map is an isomorphism. Indeed, this is the map appearing in the analogous statement in commutative algebra of the theorem we are proving, applied to $\pi_* f : \pi_* R \to \pi_* T$. But this statement of commutative algebra is not hard to prove: it follows from the universal properties and the extension-restriction of scalars adjunction. □

PROPOSITION 6.6. *Let R and T be cofibrant commutative $\mathbb{S}$-algebras, $x \in \pi_n R$ and $y \in \pi_m T$. Denote by $x \wedge y$ the image of $x \otimes y$ under the morphism*

$$\pi_* R \otimes_{\pi_* \mathbb{S}} \pi_* T \longrightarrow \pi_*(R \wedge T)\ .$$

There is a weak equivalence of commutative $\mathbb{S}$*-algebras*

$$R[x^{-1}] \wedge T[y^{-1}] \to (R \wedge T)[(x \wedge y)^{-1}].$$

Note that this is is also a map of commutative $R[x^{-1}]$ and $T[y^{-1}]$-algebras.

PROOF. Let $i_1 : R \to R \wedge T$, $i_2 : T \to R \wedge T$ be the canonical maps into the coproduct. There exists a map f making the following diagram commute.

$$\begin{array}{ccccc} R & \xrightarrow{i_1} & R \wedge T & \xrightarrow{j_{R\wedge T}} & (R \wedge T)[(x \wedge y)^{-1}] \\ {\scriptstyle j_R}\downarrow & & & \nearrow_{f} & \\ R[x^{-1}] & & & & \end{array}$$

Indeed, applying π_* to the horizontal composition, we get the map

$$\pi_*(j_{R\wedge T} \circ i_1) : \pi_* R \to \pi_*(R \wedge T)[(x \wedge y)^{-1}]$$

which maps x to $x \wedge 1$. This is an invertible element with inverse $(1 \wedge y)(x \wedge y)^{-1}$, since the map $(\pi_* i_1, \pi_* i_2) : \pi_* R \otimes_{\pi_* \mathbb{S}} \pi_* T \to \pi_*(R \wedge T)$ is multiplicative. Therefore, the property of Theorem 6.1 provides us with the arrow f in $\mathbb{S}$-**CAlg**. Similarly, we get a map $g : T[y^{-1}] \to (R \wedge T)[(x \wedge y)^{-1}]$. We assemble f and g into the coproduct map in $\mathbb{S}$-**CAlg**

$$(f, g) : R[x^{-1}] \wedge T[y^{-1}] \to (R \wedge T)[(x \wedge y)^{-1}].$$

Now recall from [**EKMM97**, Section V.1] that $R[x^{-1}]$ is weakly equivalent, in R-**Mod**, to the homotopy colimit of the tower

$$R \xrightarrow{x} \Sigma^{-n} R \xrightarrow{x} \Sigma^{-2n} R \xrightarrow{x} \dots .$$

The T-module $T[y^{-1}]$ is described similarly. The $R \wedge T$-module $(R \wedge T)[(x \wedge y)^{-1}]$ is weakly equivalent to the homotopy colimit of the tower

$$R \wedge T \xrightarrow{x \wedge y} \Sigma^{-n-m} R \wedge T \xrightarrow{x \wedge y} \Sigma^{-2n-2m} R \wedge T \xrightarrow{x \wedge y} \dots .$$

Smashing the homotopy colimit computing $R[x^{-1}]$ with the one computing $T[y^{-1}]$ we obtain the homotopy colimit computing $(R \wedge T)[(x \wedge y)^{-1}]$, since the diagonal map $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is homotopy cofinal. The map (f, g) is compatible with these identifications, hence it is a weak equivalence. □

3. Some preliminary results

3.1. Suspension spectra and THH**.** Consider the strong symmetric monoidal functor

$$\Sigma^\infty_+ : \mathbf{Top} \to \mathbb{S}\text{-}\mathbf{Mod}$$

[**EKMM97**, II.1.2]. If G is a topological commutative monoid, we denote by $\mathbb{S}[G]$ the spectrum $\Sigma^\infty_+ G$ together with the commutative $\mathbb{S}$-algebra structure induced by the monoid structure of G, as in Proposition 1.13.

Endow the category **Top** with the standard cosimplicial space $\Delta^\bullet_{\text{top}}$ and the category $\mathbb{S}$-**Mod** with the cosimplicial $\mathbb{S}$-module $\Sigma^\infty_+\Delta^\bullet_{\text{top}}$. By Theorem 3.9, these beget strong symmetric monoidal functors of geometric realization

$$|-| : s\mathbf{Top} \to \mathbf{Top} \qquad \text{and} \qquad |-| : s\mathbb{S}\text{-}\mathbf{Mod} \to \mathbb{S}\text{-}\mathbf{Mod}.$$

Recall the definition given in Chapter 3, Section 4.2: if A is a topological commutative monoid or a commutative $\mathbb{S}$-algebra, we have

$$B^{\text{cy}}(A) := |B^{\text{cy}}_\bullet(A)|.$$

It is a commutative A-algebra by Corollary 3.23. In the $\mathbb{S}$-algebra case, this object defines the topological Hochschild homology of A, denoted $THH(A)$ (which has good homotopical behavior when A is a cofibrant commutative $\mathbb{S}$-algebra [**EKMM97**, IX.2.7]).

Proposition 3.24 applied to the strong symmetric monoidal functor $\Sigma^\infty_+ : \mathbf{Top} \to \mathbb{S}\text{-}\mathbf{Mod}$ gives

PROPOSITION 6.7. *Let G be a topological commutative monoid. There is an isomorphism of commutative $\mathbb{S}[G]$-algebras*

$$THH(\mathbb{S}[G]) \xrightarrow{\cong} \mathbb{S}[B^{\text{cy}}(G)].$$

Versions of the previous proposition have appeared as Theorem 7.1 of [**HM97**] in the setting of functors with smash product and as Example 4.2.2.7 of [**DGM13**] in the setting of Γ-spaces. A version of it can already be found in [**Wal79**].

We will also need the following proposition, obtained by applying Proposition 1.56 to the strong symmetric monoidal functor $\Sigma^\infty_+ : \mathbf{Top} \to \mathbb{S}\text{-}\mathbf{Mod}$.

PROPOSITION 6.8. *Consider $G, H \in \mathbf{CMon}(\mathbf{Top})$. There is an isomorphism of commutative $\mathbb{S}[G]$-algebras*

$$\mathbb{S}[G] \wedge \mathbb{S}[H] \xrightarrow{\cong} \mathbb{S}[G \times H]$$

natural in H.

3.2. Cyclic bar construction of a topological abelian group. Let G be a topological abelian group with unit 0. Denote by BG the model for the classifying space of G which is given as the geometric realization of the reduced bar construction $B_\bullet(0, G, 0)$ of G. Therefore, BG is a topological abelian group. Moreover, if G is a CW-complex and addition is a cellular map, then the same can be said of BG. All of this is due to Milgram [**Mil67**].

The space $G \times BG$ gets the structure of a commutative G-algebra, via the inclusion of the first factor $G \to G \times BG$, which is a morphism of topological abelian groups.

PROPOSITION 6.9. *Let G be a topological abelian group. There is a homeomorphism of commutative G-algebras*

$$B^{\text{cy}}G \cong G \times BG.$$

PROOF. Let $G_\bullet$ denote the constant simplicial commutative G-algebra on G. Consider the maps $r_\bullet : B^{\text{cy}}_\bullet G \to G_\bullet$, $(g_0, \ldots, g_p) \mapsto g_0 + \cdots + g_p$, and $p_\bullet : B^{\text{cy}}_\bullet G \to B_\bullet G$, $(g_0, \ldots, g_p) \mapsto (g_1, \ldots, g_p)$. They assemble to a map

$$B^{\text{cy}}_\bullet G \xrightarrow{(r_\bullet, p_\bullet)} G_\bullet \times B_\bullet G, \quad (g_0, \ldots, g_p) \mapsto (g_0 + \cdots + g_p, g_1, \ldots, g_p).$$

We also have maps $i_\bullet : G_\bullet \to B^{\text{cy}}_\bullet G$, $g \mapsto (g, 0, \ldots, 0)$ and $s_\bullet : B_\bullet G \to B^{\text{cy}}_\bullet G$, $(g_1, \ldots, g_p) \mapsto (-g_1 - \cdots - g_p, g_1, \ldots, g_p)$. We sum them up to a map

$$G_\bullet \times B_\bullet G \xrightarrow{i_\bullet + s_\bullet} B^{\text{cy}}_\bullet G, \quad (g, g_1, \ldots, g_p) \mapsto (g - g_1 - \cdots - g_p, g_1, \ldots, g_p).$$

The maps $(r_\bullet, p_\bullet)$ and $i_\bullet + s_\bullet$ are morphisms of simplicial commutative G-algebras which are inverse to one another. (Note that the obvious isomorphisms $G \times G^p \cong G^{p+1}$ are not good, because they do not commute with the last face map.) Applying geometric realization we obtain the result. $\square$

A classical result (which we will not use) states that $B^{\text{cy}}G$ is homotopy equivalent to the free loop space of BG (see e.g. [**BHM93**, Section 2]).

3.3. Inverting an element in THH. Let R be a cofibrant commutative $\mathbb{S}$-algebra and $x \in \pi_* R$. Denote by $\eta : R \to THH(R)$ the unit. Since $THH(R) = B^{\text{cy}}(R)$ is a cofibrant commutative $\mathbb{S}$-algebra [**SVW00**, Lemma 3.6], Proposition 6.4 immediately gives a weak equivalence of commutative $R[x^{-1}]$-algebras

$$THH(R, R[x^{-1}]) \cong R[x^{-1}] \wedge_R THH(R) \xrightarrow{\sim} THH(R)[\pi_*\eta(x)^{-1}]. \tag{6.10}$$

For simplicity, denote the codomain of this arrow by $THH(R)[x^{-1}]$.

We now aim to prove that $THH(R, R[x^{-1}])$ and $THH(R[x^{-1}])$ are weakly equivalent commutative $R[x^{-1}]$-algebras. We will obtain this as a consequence of the following more general theorem, by taking the sequence (6.12) to be $\mathbb{S} \to R \to R[x^{-1}]$.

THEOREM 6.11. *Let*

$$\mathbb{S} \to A \xrightarrow{f} B \tag{6.12}$$

be a sequence of cofibrations of commutative $\mathbb{S}$-algebras. Suppose that the multiplication map $\mu : B \wedge_A B \to B$ is a weak equivalence. Then the map of commutative B-algebras

$$B \wedge_A THH(A) \cong THH(A, B) \xrightarrow{THH(f,\mathrm{id})} THH(B, B) = THH(B) \tag{6.13}$$

is a weak equivalence.

This theorem is valid *mutatis mutandis* when $\mathbb{S}$ is replaced by some cofibrant commutative $\mathbb{S}$-algebra.

We draw inspiration from [**Hön17**, Lemma 2.4.10]. For $R \in \mathbb{S}$-**CAlg**, denote $R^e := R \wedge R$.

Proof. Consider A (resp. B) as a commutative A^e-algebra (resp. B^e-algebra) via the multiplication map $A^e \to A$ (resp. $B^e \to B$). Recall that from the simplicial isomorphism of Proposition 2.23 we get an isomorphism $THH(A, B) \cong B \wedge_{A^e} B(A, A, A)$ and similarly for $THH(B)$.

Let $\tilde{B} \xrightarrow{\sim} B$ be a cofibrant replacement of B in the category of commutative B^e-algebras. There is a commutative diagram of $\mathbb{S}$-modules

$$\begin{array}{ccccc} THH(A,B) & \xleftarrow{\sim} & \tilde{B} \wedge_{A^e} B(A,A,A) & \xrightarrow{\sim} & \tilde{B} \wedge_{A^e} A \\ \downarrow{\scriptstyle THH(f,\mathrm{id})} & & \downarrow{\scriptstyle (\mathrm{id},f)} & & \downarrow{\scriptstyle \overline{f}} \\ THH(B) & \xleftarrow{\sim} & \tilde{B} \wedge_{B^e} B(B,B,B) & \xrightarrow{\sim} & \tilde{B} \wedge_{B^e} B. \end{array} \tag{6.14}$$

Recall that the two-sided bar construction $B(A, A, A)$ induces a weak equivalence of commutative A^e-algebras $B(A, A, A) \to A$ [**EKMM97**, IV.7.5] and a cofibration in $\mathbb{S}$-**CAlg** $A^e \to B(A, A, A)$ given by inclusion of the first and last smash factors. See [**Hön17**, Proof of Lemma 2.4.8] for a proof of this last fact.

The arrow (id, f) in the middle is defined via the universal property for the coproduct in commutative A^e-algebras, using the canonical map $\tilde{B} \to \tilde{B} \wedge_{B^e} B(B, B, B)$ to the first factor, and the map $B(A, A, A) \to B(B, B, B)$ defined by smash powers of f at the simplicial level followed by the canonical map to the second factor.

The arrow $\overline{f}$ is described as follows. First note that there are isomorphisms

$$\tilde{B} \wedge_{A^e} A \cong \tilde{B} \wedge_{B^e} (B^e \wedge_{A^e} A) \cong \tilde{B} \wedge_{B^e} (B \wedge_A B).$$

The last step comes from the isomorphism of commutative B^e-algebras $B^e \wedge_{A^e} A \cong B \wedge_A B$ which appears e.g. in [**Lin00**, Lemma 2.1]. Then $\overline{f}$ is defined to be the composition

$$\tilde{B} \wedge_{A^e} A \cong \tilde{B} \wedge_{B^e} (B \wedge_A B) \xrightarrow{\mathrm{id} \wedge \mu} \tilde{B} \wedge_{B^e} B.$$

The previous diagram appears as the geometric realization of a diagram in simplicial $\mathbb{S}$-modules. The arrows in this latter diagram are very explicitely defined, and it is immediate that they make the diagram commute.

Therefore, to see that $THH(f, \mathrm{id})$ is a weak equivalence, it suffices to see that $\mathrm{id} \wedge \mu$ is a weak equivalence. This is the case: indeed, the functor $\tilde{B} \wedge_{B^e} -$ preserves weak equivalences between cofibrant commutative $\mathbb{S}$-algebras because $\tilde{B}$ is cofibrant as a commutative B^e-algebra. Now note that $B \wedge_A B$ is a cofibrant commutative $\mathbb{S}$-algebra because it is a cofibrant commutative A-algebra (it is a coproduct of two cofibrant commutative A-algebras). □

Lemma 6.2 allows us to apply Theorem 6.11 to $\mathbb{S} \to R \to R[x^{-1}]$. Putting this together with the weak equivalence (6.10), we obtain:

Corollary 6.15. *Let R be a cofibrant commutative $\mathbb{S}$-algebra, and let $x \in \pi_* R$. There are weak equivalences of commutative $R[x^{-1}]$-algebras*

$$THH(R)[x^{-1}] \xleftarrow{\sim} THH(R, R[x^{-1}]) \xrightarrow{\sim} THH(R[x^{-1}]).$$

Remark 6.16. We know three proofs of the fact that Hochschild homology commutes with localizations. Weibel [**Wei94**, 9.1.8(3)] proves it using the fact that Tor behaves well under flat base change. Brylinski [**Bry89**] (see also [**Lod98**, 1.1.17]) prove it by comparing the homological functors defined on A-bimodules $S^{-1}HH_n(A, -)$ and $HH_n(S^{-1}A, S^{-1}-)$, where S is a multiplicative subset of the commutative algebra A. In [**WG91**], Geller and Weibel prove the more general result that Hochschild homology behaves well with respect to étale maps of commutative algebras $A \to B$, of which a localization map is an example. Our proof of Theorem 6.11 is closer to the first of these approaches. In a previous version of this dissertation, there was a proof closer in spirit to the proof of Brylinski, but it was more complicated and some technical aspects were unclear.

Remark 6.17. For a map $f : A \to B$ of commutative $\mathbb{S}$-algebras as in Theorem 6.11, the question of under what conditions is (6.13) a weak equivalence has been considered before. For example, in [**MM03**, Lemma 5.7] the authors prove that it holds when A and B are connective and the unit $B \to THH^A(B)$ is a weak equivalence. Mathew [**Mat17**, Theorem 1.3], working in the context of the E_∞-rings of Lurie, proved that a map $A \to B$ of E_∞-rings satisfies that (6.13) is an equivalence provided f is étale, with no hypotheses on connectivity. There is a notion of localization of E_∞-rings, and Lurie proved that localization maps are étale [**Lur**, 7.5.1.13]. This gives a one-line proof of Theorem 6.11 applied to $\mathbb{S} \to R \to R[x^{-1}]$ in the context of E_∞-rings.

4. Topological Hochschild homology of KU

4.1. Topological Hochschild homology of $\mathbb{S}[G][x^{-1}]$. Let G be a topological abelian group which is a CW-complex with a cellular addition map. As remarked in Section 3.2, this assumption guarantees that BG is again a CW-complex with a cellular multiplication map.

Let $x \in \pi_n \mathbb{S}[G]$. We prove the following theorem below.

THEOREM 6.18. *The commutative* $\mathbb{S}[G][x^{-1}]$*-algebras* $THH(\mathbb{S}[G][x^{-1}])$ *and* $\mathbb{S}[G][x^{-1}][BG]$ *are weakly equivalent as* $\mathbb{S}[G][x^{-1}]$*-algebras.*

For any commutative $\mathbb{S}$-algebra A, the notation $A[BG]$ stands for the commutative A-algebra $A \wedge \mathbb{S}[BG]$: thus, its underlying A-module is $A \wedge (BG)_+$. No confusion should arise from the usage of square brackets for two different notions.

We first isolate the part of the proof that does not involve inverting an element.

LEMMA 6.19. *There is an isomorphism of commutative* $\mathbb{S}[G]$*-algebras*

$$THH(\mathbb{S}[G]) \cong \mathbb{S}[G] \wedge \mathbb{S}[BG] = \mathbb{S}[G][BG].$$

PROOF. It is an application of Propositions 6.7, 6.9 and 6.8, in that order:

$$THH(\mathbb{S}[G]) \xrightarrow{\cong} \mathbb{S}[B^{\mathrm{cy}}G] \xrightarrow{\cong} \mathbb{S}[G \times BG] \xleftarrow{\cong} \mathbb{S}[G] \wedge \mathbb{S}[BG]. \qquad \square$$

PROOF OF THEOREM 6.18. By Corollary 6.15, we obtain a zig-zag of two weak equivalences of commutative $\mathbb{S}[G][x^{-1}]$-algebras

$$THH(\mathbb{S}[G][x^{-1}]) \simeq THH(\mathbb{S}[G])[x^{-1}].$$

Lemma 6.19 gives an isomorphism $THH(\mathbb{S}[G]) \cong \mathbb{S}[G] \wedge \mathbb{S}[BG]$ such that

$$THH(\mathbb{S}[G])[x^{-1}] \cong (\mathbb{S}[G] \wedge \mathbb{S}[BG])[(x \wedge 1)^{-1}].$$

Now Proposition 6.6 gives a weak equivalence of commutative $\mathbb{S}[G][x^{-1}]$-algebras

$$(\mathbb{S}[G] \wedge \mathbb{S}[BG])[(x \wedge 1)^{-1}] \simeq \mathbb{S}[G][x^{-1}] \wedge \mathbb{S}[BG] = \mathbb{S}[G][x^{-1}][BG]$$

finishing the proof. $\square$

4.2. Snaith's theorem. There is a commutative $\mathbb{S}$-algebra KU of complex topological K-theory [**EKMM97**, VIII.4.3]. It is obtained by applying the localization theorem we reviewed in Theorem 6.1 to the cofibrant commutative $\mathbb{S}$-algebra ku of connective K-theory and its Bott element. Here ku is constructed by multiplicative infinite loop space theory.

The presentation for KU which we will use is given by the following version of a theorem of Snaith [**Sna79**], [**Sna81**]:

THEOREM 6.20. KU *is weakly equivalent to the cofibrant commutative* $\mathbb{S}$*-algebra* $\mathbb{S}[\mathbb{C}P^\infty][x^{-1}]$*, where* $x \in \pi_2(\mathbb{S}[\mathbb{C}P^\infty])$ *is represented by the map induced from the inclusion* $\mathbb{C}P^1 \to \mathbb{C}P^\infty$*, i.e.*

$$\Sigma^\infty S^2 \cong \Sigma^\infty \mathbb{C}P^1 \to \mathbb{S} \vee \Sigma^\infty \mathbb{C}P^\infty \simeq \Sigma^\infty_+ \mathbb{C}P^\infty.$$

REMARK 6.21. We thank Christian Schlichtkrull for pointing out the article [**Art83**] to us. In Theorems 5.1 and 5.2 therein, it is proven that if $t \in \pi_n(\mathbb{S}[K(\mathbb{Z}, n)])$ is a generator, then $\mathbb{S}[K(\mathbb{Z}, n)][t^{-1}]$ is contractible for n odd and is equivalent to $H\mathbb{Q}[t^{\pm 1}]$ for $n \geq 4$ even. So the case $n = 2$ which we treat here is the only interesting localization of $\mathbb{S}[K(\mathbb{Z}, n)]$.

4.3. Rationalization. In this short section we review some facts about rationalization that we will be using.

If X is an $\mathbb{S}$-module, we denote by $X_\mathbb{Q}$ its rationalization. Our model for $X_\mathbb{Q}$ is given by $H\mathbb{Q} \wedge X$, where $H\mathbb{Q}$ is any model for the Eilenberg-Mac Lane commutative $\mathbb{S}$-algebra of $\mathbb{Q}$ which is cofibrant as an $\mathbb{S}$-module. Therefore, the rationalization functor is $H\mathbb{Q} \wedge -$, and as such it is lax symmetric monoidal. The structure map $X_\mathbb{Q} \wedge Y_\mathbb{Q} \to (X \wedge Y)_\mathbb{Q}$ is a weak equivalence when X and Y are cofibrant $\mathbb{S}$-modules, since the multiplication of $H\mathbb{Q}$ is a weak equivalence. Note that we do not need to derive the functor $H\mathbb{Q} \wedge -$, since $H\mathbb{Q}$ is cofibrant.

Let n be any integer. The degree n map $n : \mathbb{S} \to \mathbb{S}$ induces a map $n : X \to X$ on any $\mathbb{S}$-module X. If $p : X \to X$ is a weak equivalence for every prime p then the homotopy groups of X are rational, since p induces the multiplication by p map on homotopy groups. Therefore X is rational, i.e. the rationalization map $X \to X_\mathbb{Q}$ is an equivalence.

We will also need the following fact concerning the rationalization of Eilenberg-Mac Lane spaces [**FHT01**, Page 202]: for $n \geq 2$,

$$K(\mathbb{Z}, n)_\mathbb{Q} \simeq \begin{cases} S^n_\mathbb{Q} & \text{if } n \text{ is odd,} \\ \Omega S^{n+1}_\mathbb{Q} & \text{if } n \text{ is even.} \end{cases}$$

Actually, the authors prove that for n even, $\Omega S^{n+1} \to K(\mathbb{Z}, n)$ is a rational equivalence, so that $K(\mathbb{Z}, n)_\mathbb{Q} \simeq (\Omega S^{n+1})_\mathbb{Q}$, which is not exactly what we wrote. But indeed, for any simply connected space X we have that $(\Omega X)_\mathbb{Q} \simeq \Omega(X_\mathbb{Q})$. This follows from comparing the rational cohomology Eilenberg-Moore spectral sequences [**McC01**, Corollary 7.16] for ΩX and for $\Omega X_\mathbb{Q}$ via the rationalization map $X \to X_\mathbb{Q}$: we obtain that the map $\Omega X \to \Omega X_\mathbb{Q}$ is a rationalization map, for $\Omega X_\mathbb{Q}$ is rational. Symbolically,

$$\begin{array}{ccc} \operatorname{Tor}_{H^*(X;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) & \Longrightarrow & H^*(\Omega X; \mathbb{Q}) \\ \cong \big\downarrow & & \big\downarrow \cong \\ \operatorname{Tor}_{H^*(X_\mathbb{Q};\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) & \Longrightarrow & H^*(\Omega X_\mathbb{Q}, \mathbb{Q}). \end{array}$$

4.4. The main results. As a particular case of Theorem 6.18, we have:

THEOREM 6.22. *The commutative KU-algebras $THH(KU)$ and $KU[K(\mathbb{Z}, 3)]$ are weakly equivalent KU-algebras.*

REMARK 6.23. Compare with what happens to $THH(MU)$: in [**BCS10**], the authors establish an equivalence of $\mathbb{S}$-modules $THH(MU) \simeq MU \wedge SU_+$. They actually prove the following more general result. Let BF denote the classifying space for stable spherical fibrations. If $f : X \to BF$ is a 3-fold loop map and $T(f)$ is its Thom spectrum, then there is a weak equivalence of $\mathbb{S}$-modules

$$THH(T(f)) \simeq T(f) \wedge BX_+. \tag{6.24}$$

Note that this result was improved to an equivalence of E_∞ $\mathbb{S}$-algebras by Schlichtkrull [**Sch11**, Corollary 1.2] in the case where X is a grouplike E_∞-space and f is an E_∞-map.

Our Theorem 6.22 gives in particular a weak equivalence of $\mathbb{S}$-modules $THH(KU) \simeq KU \wedge K(\mathbb{Z}, 3)_+$: by comparing this formula to (6.24), one is naturally led to conjecture that KU is the Thom spectrum of an ∞-loop map $K(\mathbb{Z}, 2) \simeq BU(1) \to BU$. However, this is not possible, since Thom spectra are connective. On the other hand, Sagave and Schlichtkrull [**SS14**] have introduced *graded* Thom spectra, and these can be non-connective. It also seems unlikely that KU will be the graded Thom spectrum of a map $BU(1) \to BU \times \mathbb{Z}$, since the image will be contained in one of the components of $BU \times \mathbb{Z}$. We would like to understand why does KU behave like a Thom spectrum, at least to the eyes of topological Hochschild homology.

We will now describe the commutative KU-algebra $THH(KU)$ as the free commutative KU-algebra on the KU-module $\Sigma KU_{\mathbb{Q}}$, and we will prove this free commutative algebra to be weakly equivalent to the split square-zero extension of KU by $\Sigma KU_{\mathbb{Q}}$. Let us first define this concept.

Let R be a commutative $\mathbb{S}$-algebra, let A be a commutative R-algebra and let M be a non-unital commutative A-algebra. Then $A \vee M$ (coproduct of A-modules) has a commutative A-algebra structure. Indeed, after distributing, a multiplication map

$$(A \vee M) \wedge_A (A \vee M) \to A \vee M$$

looks like

$$(A \wedge_A A) \vee (A \wedge_A M) \vee (M \wedge_A A) \vee (M \wedge_A M) \to A \vee M. \tag{6.25}$$

We may define a map like (6.25) by defining maps from each of the wedge summands to $A \vee M$. Define the maps to $A \vee M$ from $A \wedge_A A$, $A \wedge_A M$ and $M \wedge_A A$ to be the canonical isomorphisms followed by the canonical maps into the respective factor. Finally, consider the map $M \wedge_A M \to A \vee M$ given by the multiplication map of M followed by the canonical map to $A \vee M$. We have thus defined a multiplication map (6.25) such that $A \vee M$ is a commutative A-algebra with unit given by the canonical map $A \to A \vee M$. We say that $A \vee M$ is a *split extension of A by M*. If the multiplication of M is trivial, then $A \vee M$ is a *split square-zero extension of A by M*; in this case, M is no more than an A-module.

Conversely, if A is a commutative R-algebra with augmentation $\varepsilon : A \to R$, then there is a splitting in commutative augmented R-algebras $A \simeq R \vee \overline{A}$ where $\overline{A}$ is a non-unital commutative R-algebra fitting into a fiber sequence

$$\overline{A} \to A \to R.$$

More precisely, the underlying R-module of $\overline{A}$ fits into the following pullback square in R-**Mod**,

$$\begin{array}{ccc} \overline{A} & \xrightarrow{i} & A \\ \downarrow & & \downarrow \varepsilon \\ 0 & \longrightarrow & R \end{array}$$

and it gets a non-unital multiplication from the universal property of pullbacks, by considering the following commutative diagram in R-**Mod**. See [**Bas99**, Section 2] for further elaboration.

$$\begin{array}{ccccc} \overline{A} \wedge_R \overline{A} & \xrightarrow{i \wedge i} & A \wedge_R A & \xrightarrow{\mu} & A \\ \downarrow & & \downarrow \varepsilon\wedge\varepsilon & & \downarrow \varepsilon \\ 0 & \longrightarrow & R \wedge_R R & \xrightarrow[\cong]{} & R \end{array}$$

In particular, there is a splitting of commutative augmented A-algebras

$$THH(A) \simeq A \vee \overline{THH}(A). \tag{6.26}$$

The rest of this section is devoted to the proof of the following

THEOREM 6.27. *There is a morphism of commutative augmented KU-algebras*

$$\tilde{f} : F(\Sigma KU_{\mathbb{Q}}) \to THH(KU)$$

*which is a weak equivalence. Here $F : KU$-**Mod** $\to KU$-**CAlg** is the free commutative algebra functor.*

Moreover, $F(\Sigma KU_{\mathbb{Q}})$ is weakly equivalent as an augmented commutative KU-algebra to the split square-zero extension $KU \vee \Sigma KU_{\mathbb{Q}}$.

The morphism $\tilde{f}$ is obtained by the universal property of F from a map of KU-modules $f : \Sigma KU_{\mathbb{Q}} \to THH(KU)$ to be described below (6.36).

REMARK 6.28. The functor F, or more generally, the free commutative algebra functor $F_R : R$-**Mod** $\to R$-**CAlg** where R is a commutative $\mathbb{S}$-algebra, is the left adjoint of the forgetful functor $U_R : R\text{-}\mathbf{CAlg} \to R\text{-}\mathbf{Mod}$, or alternatively, the free algebra functor for the monad $\mathbb{P}_R$ on R-**Mod** defined as

$$\mathbb{P}_R(M) = \bigvee_{n\geq 0} M^{\wedge_R n}/\Sigma_n = R \vee M \vee \bigvee_{n\geq 2} M^{\wedge_R n}/\Sigma_n, \tag{6.29}$$

where Σ_n is the symmetric group on n elements (see e.g. [**EKMM97**, II.7.1] or [**Bas99**, Section 1]). Note that $F_R M$ is augmented over R: the augmentation is the projection on the 0-th term.

As explained in Section 1, the functor $U_R : R\text{-}\mathbf{CAlg} \to R\text{-}\mathbf{Mod}$ is a right Quillen functor, so $F_R : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{CAlg}$ is a left Quillen functor. In particular, it preserves weak equivalences between cofibrant R-modules.

Note as well that, if $M \in R\text{-}\mathbf{Mod}$ is cofibrant, then the arrow $\bigvee_{n\geq 0}(M^{\wedge_R n})_{h\Sigma_n} \to F_R(M)$ induced from the canonical arrows from the homotopy orbits to the orbits is a weak equivalence [**EKMM97**, Theorem III.5.1]. This is a step in the proof of the determination of the model structure on R-**CAlg**.

REMARK 6.30. A spectrum-level result related to Theorem 6.27 was obtained by McClure and Staffeldt in [**MS93**, Theorem 8.1]: they showed that $THH(L) \simeq L \vee \Sigma L_{\mathbb{Q}}$ as spectra, where L is the p-adic completion of the Adams summand of KU for a given odd prime p; the result was extended to $p = 2$ by Angeltveit, Hill and Lawson in [**AHL10**, 2.3]. Ausoni [**Aus05**, Proposition 7.13] formulates without proof the analogous theorem (for an odd p) for KU completed at p in place of L. In Corollary 7.9 of [**AHL10**], the authors show that $THH(KO) \simeq KO \vee \Sigma KO_{\mathbb{Q}}$ as KO-modules. The methods used in the proofs of the results just cited are different from ours.

We first prove a couple of results needed for the proof. Note that in the following statement we are considering $K(\mathbb{Z}, 3)$ as a pointed space: we are not adding a disjoint basepoint.

PROPOSITION 6.31. *There is a weak equivalence $KU \wedge K(\mathbb{Z}, 3) \simeq \Sigma KU_{\mathbb{Q}}$ of KU-modules.*

PROOF. Let p be a prime and consider the cofiber sequence of KU-modules

$$KU \xrightarrow{p} KU \longrightarrow KU/p \longrightarrow \Sigma KU. \tag{6.32}$$

If $p > 2$, then KU/p is equivalent to $\bigvee_{i=0}^{p-2} \Sigma^{2i} K(1)$ (see [**Ada69**, Lecture 4]), where $K(1) \simeq L/p$ is the first Morava K-theory at p. If $p = 2$, then $K(1) \simeq KU/2$.

The homology $K(1)_* K(\mathbb{Z}, 3)$ is trivial: see [**RW80**, Theorem 12.1] for the $p > 2$ case, and [**JW85**, Appendix] for the $p = 2$ case. Therefore, after smashing (6.32) with $K(\mathbb{Z}, 3)$, we get a weak equivalence of KU-modules

$$KU \wedge K(\mathbb{Z},3) \xrightarrow[p\wedge \mathrm{id}]{\sim} KU \wedge K(\mathbb{Z},3)$$

for all primes p. This means that $KU \wedge K(\mathbb{Z},3)$ is rational, and so

$$KU \wedge K(\mathbb{Z},3) \simeq (KU \wedge K(\mathbb{Z},3))_{\mathbb{Q}} \simeq KU_{\mathbb{Q}} \wedge K(\mathbb{Z},3)_{\mathbb{Q}} \simeq KU_{\mathbb{Q}} \wedge S^3_{\mathbb{Q}} \simeq \Sigma KU_{\mathbb{Q}}$$

by the results quoted in Section 4.3, plus Bott periodicity for the last step. □

PROPOSITION 6.33. *Let R be a commutative $\mathbb{S}$-algebra and $F_R : R$-**Mod** $\to R$-**CAlg** be the free commutative algebra functor. The augmented commutative R-algebra $F_R(\Sigma R_{\mathbb{Q}})$ is weakly equivalent to the split square-zero extension $R \vee \Sigma R_{\mathbb{Q}}$.*

REMARK 6.34. Note that we are applying F_R to a cofibrant R-module. Indeed, since $H\mathbb{Q}$ is a cofibrant $\mathbb{S}$-module and S^1 is a cofibrant based space, then $S^1 \wedge H\mathbb{Q}$ is a cofibrant $\mathbb{S}$-module. Now, the extension of scalars functor $R\wedge - : \mathbb{S}$-**Mod** $\to R$-**Mod** is left Quillen: indeed, its right adjoint, the forgetful functor, is right Quillen since the model structure in R-**Mod** is created through it. Therefore, $R \wedge (S^1 \wedge H\mathbb{Q}) \cong \Sigma R_{\mathbb{Q}}$ is a cofibrant R-module.

PROOF. Recall Remark 6.28 describing the functor F_R. Note that for an $\mathbb{S}$-module X, we have a natural isomorphism $F_R(R \wedge X) \cong R \wedge F_{\mathbb{S}}(X)$. Indeed,

$$F_R(R\wedge X) = \bigvee_{n\geq 0} (R\wedge X)^{\wedge_R n}/\Sigma_n \cong R \wedge \bigvee_{n\geq 0} X^{\wedge n}/\Sigma_n = R \wedge F_{\mathbb{S}}(X)$$

since the left adjoint functor $R \wedge - : \mathbb{S}$-**Mod** $\to R$-**Mod** preserves colimits. Therefore, $F_R(\Sigma R_{\mathbb{Q}}) = F_R(R \wedge S^1_{\mathbb{Q}}) \cong R \wedge F_{\mathbb{S}}(S^1_{\mathbb{Q}})$. We have

$$F_{\mathbb{S}}(S^1_{\mathbb{Q}}) = \mathbb{S} \vee S^1_{\mathbb{Q}} \vee \bigvee_{n\geq 2} (S^1_{\mathbb{Q}})^{\wedge n}/\Sigma_n \simeq \mathbb{S} \vee S^1_{\mathbb{Q}} \vee \bigvee_{n\geq 2} \Big((S^1)^{\wedge n}/\Sigma_n\Big)_{\mathbb{Q}}$$

where we have used that the rationalization functor $H\mathbb{Q} \wedge -$ commutes with colimits and that the rationalization of a smash product of finitely many factors is weakly equivalent to the smash product of the rationalizations.

Now we claim that $(S^1)^{\wedge n}/\Sigma_n$ is contractible for $n \geq 2$, and this finishes the proof. To see this, let X be a based space and consider $\widetilde{SP}^n(X) := X^{\wedge n}/\Sigma_n$ and $SP^n(X) := X^{\times n}/\Sigma_n$. There is a map $f_n : SP^{n-1}(X) \to SP^n(X)$ given by inserting a basepoint, and its cofiber is $\widetilde{SP}^n(X)$. When $X = S^1$, the map f_n is a homotopy equivalence for any $n \geq 2$ [**AGP02**, 5.2.23], so that $\widetilde{SP}^n(S^1)$ is contractible for all $n \geq 2$. □

PROOF OF THEOREM 6.27. First, we work additively, and then we will determine the multiplicative structure.

Recall that for any based space X, there is a weak equivalence $\Sigma^\infty_+ X \simeq \mathbb{S} \vee \Sigma^\infty X$ coming from the homotopy equivalence of spaces $\Sigma(X_+) \simeq S^1 \vee \Sigma X$. Combining this with Theorem 6.22 and Proposition 6.31, we obtain weak equivalences of KU-modules

$$\begin{aligned} THH(KU) &\simeq KU \wedge \Sigma^\infty_+ K(\mathbb{Z},3) \simeq KU \wedge (\mathbb{S} \vee \Sigma^\infty K(\mathbb{Z},3)) \cong \\ &\simeq KU \vee (KU \wedge \Sigma^\infty K(\mathbb{Z},3)) \simeq KU \vee \Sigma KU_{\mathbb{Q}}. \end{aligned}$$

This splitting is compatible with the splitting (6.26) since both are just splitting off the unit, so we have a weak equivalence of KU-modules $\Sigma KU_{\mathbb{Q}} \xrightarrow{\sim} \overline{THH}(KU)$.

In particular, we have an isomorphism

$$\pi_* THH(KU) \cong \mathbb{Z}[t^{\pm 1}] \oplus \Sigma \mathbb{Q}[t^{\pm 1}] \tag{6.35}$$

of $\mathbb{Z}[t^{\pm 1}]$-graded modules.

We will now determine the multiplicative structure. Consider the map of KU-modules

$$f:\ \Sigma KU_{\mathbb{Q}} \xrightarrow{\ \sim\ } \overline{THH}(KU) \longrightarrow THH(KU). \tag{6.36}$$

In homotopy groups, this is an isomorphism $\Sigma\mathbb{Q}[t^{\pm 1}] \to \pi_*\overline{THH}(KU)$ followed by the inclusion into $\pi_* THH(KU)$.

By the universal property satisfied by F, we get that f induces a map of commutative KU-algebras

$$\tilde{f} : F(\Sigma KU_{\mathbb{Q}}) \to THH(KU).$$

Note that, by definition of f, we have that $\varepsilon \circ f$ is the trivial map, where $\varepsilon : THH(KU) \to KU$ is the augmentation. This implies that $\tilde{f}$ preserves the augmentation.

Now, $\tilde{f}$ is a weak equivalence. Indeed, after identifying $F(\Sigma KU_{\mathbb{Q}})$ with the split square-zero extension $KU \vee \Sigma KU_{\mathbb{Q}}$ (Proposition 6.33), $\tilde{f}$ amounts to the map of commutative KU-algebras $(\eta, f) : KU \vee \Sigma KU_{\mathbb{Q}} \to THH(KU)$. But (η, f) is a weak equivalence by construction of f. □

Corollary 6.37. *The map*

$$(\eta, f) : KU \vee \Sigma KU_{\mathbb{Q}} \to THH(KU)$$

is a weak equivalence of augmented commutative KU-algebras, where $\eta : KU \to THH(KU)$ is the unit, the map f was defined in (6.36), and $KU \vee \Sigma KU_{\mathbb{Q}}$ is a split square-zero extension.

4.5. The morphism σ. If R is a commutative $\mathbb{S}$-algebra, there is a natural transformation of $\mathbb{S}$-modules [**MS93**, Section 3], [**EKMM97**, IX.3.8], [**AR05**, 3.12]

$$\sigma : \Sigma R \to THH(R).$$

Consider the map

$$(\eta, \sigma) : KU \vee \Sigma KU \to THH(KU).$$

It is tempting to conjecture that its rationalization

$$(\eta_{\mathbb{Q}}, \sigma_{\mathbb{Q}}) : KU_{\mathbb{Q}} \vee \Sigma KU_{\mathbb{Q}} \to THH(KU)_{\mathbb{Q}}$$

is a weak equivalence, since by the results of the previous section, the $\mathbb{S}$-modules $KU_{\mathbb{Q}} \vee \Sigma KU_{\mathbb{Q}}$ and $THH(KU)_{\mathbb{Q}}$ are weakly equivalent.

However, this is not the case. I thank Geoffroy Horel and Thomas Nikolaus for pointing out this fact and the following proof to me. We will prove that $\sigma : \Sigma KU \to THH(KU)$ is zero in π_1, therefore it is still zero after rationalization. By naturality of σ, we have a commutative diagram

$$\begin{array}{ccc} \Sigma\mathbb{S} & \xrightarrow{\sigma} & THH(\mathbb{S}) \simeq \mathbb{S} \\ \downarrow{\scriptstyle \Sigma\iota} & & \downarrow{\scriptstyle THH(\iota)} \\ \Sigma KU & \xrightarrow[\sigma]{} & THH(KU) \end{array} \tag{6.38}$$

where ι is the unit of KU. After taking π_1, we obtain a commutative diagram of abelian groups

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Q}. \end{array} \tag{6.39}$$

Therefore, $\mathbb{Z} \to \mathbb{Q}$ must be the zero map, since only the abelian group map $\mathbb{Z}/2 \to \mathbb{Q}$ is the zero map.

Note that the same proof works for L (the p-adic completion of the Adams summand of KU, p a prime) instead of KU. Recall that $\pi_* L \cong \mathbb{Z}_{(p)}[(v_1)^{\pm 1}]$, with v_1 in degree $2p-2$. After replacing KU with L in (6.38) and taking π_1, we obtain a square which looks like (6.39) except with a $\mathbb{Z}_{(p)}$ on the lower left corner. The vertical map $\mathbb{Z} \to \mathbb{Z}_{(p)}$ is the unit of $\mathbb{Z}_{(p)}$: this still forces $\pi_1\sigma : \pi_1(\Sigma L) \to \pi_1(THH(L))$ to be zero.

This corrects an error in [**MS93**, 8.4] where it is claimed that there is a weak equivalence $L_{\mathbb{Q}} \vee \Sigma L_{\mathbb{Q}} \xrightarrow{\sim} THH(L)_{\mathbb{Q}}$ induced by (η, σ). As a positive result, we have Corollary 6.37 instead.

5. Iterated topological Hochschild homology of *KU*

Let A be a commutative $\mathbb{S}$-algebra. We denote by $THH^n(A)$ the *iterated topological Hochschild homology of* A, i.e. $THH(\ldots(THH(A)))$ where THH is applied n times. Other expressions for $THH^n(A)$ include $T^n \otimes A$ or $\Lambda_{T^n}(A)$, where T^n is an n-torus and Λ is the Loday functor [**CDD11**].

We will now give two different descriptions of $THH^n(KU)$ for $n \geq 2$. The first one, given in Corollary 6.45, directly generalizes Theorem 6.22. We have also given a description of the commutative KU-algebra $THH(KU)$ as a split square-zero extension in Theorem 6.27. For $n \geq 2$, $THH^n(KU)$ is not a split square-zero extension of KU, as we shall see. However, it is a split extension: we will describe the non-unital commutative algebra structure of the homotopy groups of its augmentation ideal, which is rational as in the $n = 1$ case.

5.1. Description via Eilenberg-Mac Lane spaces. Let G be a topological abelian group which is a CW-complex with a cellular addition map. Applying Lemma 6.19 and Proposition 6.8, we obtain isomorphisms of commutative $\mathbb{S}[G]$-algebras:

$$\begin{aligned} THH^2(\mathbb{S}[G]) &\cong THH(\mathbb{S}[G] \wedge \mathbb{S}[BG]) \cong THH(\mathbb{S}[G \times BG]) \\ &\cong \mathbb{S}[G \times BG] \wedge \mathbb{S}[B(G \times BG)] \cong \mathbb{S}[G] \wedge \mathbb{S}[BG \times BG \times B^2G] \end{aligned}$$

which we have written as $\mathbb{S}[G][BG \times BG \times B^2G]$.

For general n, the same type of computation gives a description of $THH^n(\mathbb{S}[G])$: we obtain an isomorphism of commutative $\mathbb{S}[G]$-algebras

$$THH^n(\mathbb{S}[G]) \cong \mathbb{S}[G][B^{a_1}G \times \cdots \times B^{a_{2^n-1}}G]. \tag{6.40}$$

The numbers a_i can be described as follows. Let $v_0 = 0$. Define by induction

$$v_n = (v_{n-1}, v_{n-1} + (1, \ldots, 1)) = (a_0, \ldots, a_{2^n-1}) \in \mathbb{N}^{2^n} \tag{6.41}$$

for $n \geq 1$. For example, $v_1 = (0, 1)$, $v_2 = (0, 1, 1, 2)$ and $v_3 = (0, 1, 1, 2, 1, 2, 2, 3)$. This sequence of integers can be found in the On-Line Encyclopedia of Integer Sequences [**Slo**]. We can give an easier description. Let I_n be the multiset having as elements the numbers i with multiplicity $\binom{n}{i}$, for $i = 1, \ldots, n$. Denote the multiplicity of an element x of a multiset by $|x|$. Now note that the multiset underlying the sequence $(a_1, \ldots, a_{2^n-1})$ defined in (6.41) coincides with I_n, by Pascal's rule. Therefore, the isomorphism (6.40) can be reformulated as

$$THH^n(\mathbb{S}[G]) \simeq \mathbb{S}[G]\left[\prod_{i=1}^{n} (B^iG)^{\times\binom{n}{i}}\right]. \tag{6.42}$$

The following theorem generalizes Theorem 6.18 to higher iterations of THH.

Theorem 6.43. *Let $x \in \pi_*\mathbb{S}[G]$. There is a zig-zag of weak equivalences of commutative $\mathbb{S}[G][x^{-1}]$-algebras*

$$THH^n(\mathbb{S}[G][x^{-1}]) \simeq \mathbb{S}[G][x^{-1}][B^{a_1}G \times \cdots \times B^{a_{2^n-1}}G],$$

or alternatively,

$$THH^n(\mathbb{S}[G][x^{-1}]) \simeq \mathbb{S}[G][x^{-1}]\left[\prod_{i=1}^{n} (B^iG)^{\times\binom{n}{i}}\right].$$

PROOF. The proof is by induction. The base case is Theorem 6.18. We do the induction step for $n = 2$ for simplicity: for higher n it is analogous, only with more indices to juggle around. By Theorem 6.18, there is a zig-zag of weak equivalences of commutative $\mathbb{S}[G][x^{-1}]$-algebras

$$THH^2(\mathbb{S}[G][x^{-1}]) = THH(THH(\mathbb{S}[G][x^{-1}])) \simeq THH(\mathbb{S}[G][x^{-1}][BG]).$$

Applying Propositions 6.6 and 6.8, we get

$$\begin{aligned}\mathbb{S}[G][x^{-1}][BG] &= \mathbb{S}[G][x^{-1}] \wedge \mathbb{S}[BG] \simeq (\mathbb{S}[G] \wedge \mathbb{S}[BG])[(x \wedge 1)^{-1}] \\ &\cong S[G \times BG][(x, e)^{-1}] \end{aligned} \tag{6.44}$$

where e is the unit of BG. Continuing the computation, we apply (6.44), Theorem 6.18 and (6.44) again, obtaining:

$$\begin{aligned} THH(\mathbb{S}[G][x^{-1}][BG]) &\simeq THH(\mathbb{S}[G \times BG][(x,e)^{-1}]) \\ &\simeq \mathbb{S}[G \times BG][(x,e)^{-1}][B(G \times BG)] \\ &\simeq (\mathbb{S}[G][x^{-1}] \wedge \mathbb{S}[BG])[BG \times B^2G] \\ &\cong \mathbb{S}[G][x^{-1}][BG \times BG \times B^2G]. \end{aligned}$$

□

COROLLARY 6.45. *There is a weak equivalence of commutative KU-algebras*

$$THH^n(KU) \simeq KU[K(\mathbb{Z}, a_1 + 2) \times \cdots \times K(\mathbb{Z}, a_{2^n-1} + 2)], \tag{6.46}$$

or alternatively,

$$THH^n(KU) \simeq KU\left[\prod_{i=1}^{n} K(\mathbb{Z}, i+2)^{\times \binom{n}{i}}\right]. \tag{6.47}$$

For example,

$$THH^2(KU) \simeq KU[K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 4)]. \tag{6.48}$$

5.2. The augmentation ideal. We first need a generalization of Proposition 6.31:

PROPOSITION 6.49. *Let $r \geq 3$. There are weak equivalences of KU-modules*

$$KU \wedge K(\mathbb{Z}, r) \simeq \begin{cases} \Sigma KU_{\mathbb{Q}} & \text{if } r \text{ is odd,} \\ \bigvee_{m \geq 1} KU_{\mathbb{Q}} & \text{if } r \text{ is even.} \end{cases}$$

PROOF. When r is odd, the proof of Proposition 6.31 works just as well, and when r is even it gives us

$$KU \wedge K(\mathbb{Z}, r) \simeq KU_{\mathbb{Q}} \wedge K(\mathbb{Z}, r)_{\mathbb{Q}}.$$

So let r be even. As noted in Section 4.3, $K(\mathbb{Z}, r)_{\mathbb{Q}} \simeq \Omega S^{r+1}_{\mathbb{Q}}$. Now we use the James splitting which says that, for X a connected pointed CW-complex, $\Sigma\Omega\Sigma X \simeq \Sigma \bigvee_{m \geq 1} X^{\wedge m}$.

Therefore, $\Sigma^\infty \Omega\Sigma X \simeq \Sigma^\infty \bigvee_{m\geq 1} X^{\wedge m}$. Rationalizing it and applying it to $X = S^r$, we obtain

$$\Sigma^\infty K(\mathbb{Z}, r)_{\mathbb{Q}} \simeq \Sigma^\infty \Omega S^{r+1}_{\mathbb{Q}} \simeq \Sigma^\infty \bigvee_{m\geq 1} S^{rm}_{\mathbb{Q}}.$$

Since r is even, Bott periodicity gives the result. □

COROLLARY 6.50. *The augmentation ideal* $\overline{THH}^n(KU)$ *is rational.*

PROOF. The expression (6.46) gives, after splitting off the units of the spherical group rings, a weak equivalence of KU-modules

$$(6.51)\quad THH^n(KU) \simeq KU \wedge (\mathbb{S} \vee \Sigma^\infty[K(\mathbb{Z}, a_1 + 2)]) \wedge \cdots \wedge (\mathbb{S} \vee \Sigma^\infty[K(\mathbb{Z}, a_{2^n-1} + 2)]).$$

Observe that if T is a rational $\mathbb{S}$-module and X is any $\mathbb{S}$-module, then $T \wedge X \simeq T \wedge X_{\mathbb{Q}}$. Distributing the terms in the previous expression and applying Proposition 6.49 gives the result. □

We can be more explicit about the additive structure by expanding (6.51). For example, for $n = 2$, we get a weak equivalence of KU-modules

$$(6.52)\quad THH^2(KU) \simeq KU \vee (\Sigma KU_{\mathbb{Q}})^{\vee 2} \vee KU_{\mathbb{Q}} \vee \left(\bigvee_{m\geq 1} \Sigma KU_{\mathbb{Q}}\right)^{\vee 2} \vee \left(\bigvee_{m\geq 1} KU_{\mathbb{Q}}\right)^{\vee 2}.$$

In general, $THH^n(KU)$ is a wedge of 2^{2^n-1} terms, only one of them being KU (it is the term corresponding to $KU \wedge \mathbb{S} \wedge \cdots \wedge \mathbb{S}$ in (6.51)), and the other factors are of the form

$$KU \wedge \bigwedge_{\substack{i=1\\ i \text{ odd}}}^{s} K(\mathbb{Z}, i) \wedge \bigwedge_{\substack{j=1\\ j \text{ even}}}^{t} K(\mathbb{Z}, j)$$

which are weakly equivalent to either

$$\begin{cases} \bigvee_{j=1}^{t} \bigvee_{m\geq 1} \Sigma KU_{\mathbb{Q}} & \text{if } s \text{ is odd, or} \\ \bigvee_{j=1}^{t} \bigvee_{m\geq 1} KU_{\mathbb{Q}} & \text{if } s \text{ is even.} \end{cases}$$

This expression for $THH^n(KU)$ as a wedge of KU and a $KU_{\mathbb{Q}}$-module is not a split square-zero extension, as we will now see.

5.2.1. *The homotopy algebra* $\overline{THH}^n_*(KU)$. From what we have just observed, we have that $\overline{THH}^n(KU)$ is a non-unital commutative $KU_{\mathbb{Q}}$-algebra. Thus, its homotopy groups are a non-unital commutative $\mathbb{Q}[t^{\pm 1}]$-algebra, which we now aim to describe.

Since THH commutes with rationalization, we get a weak equivalence

$$\overline{THH}^n(KU) \xrightarrow{\sim} \overline{THH}^n(KU_{\mathbb{Q}})$$

of non-unital commutative KU-algebras. We aim to describe

$$\overline{THH}^n_*(KU) \cong \overline{THH}^n_*(KU_{\mathbb{Q}}).$$

We will describe the latter. To do so, we look at $THH^n_*(KU_{\mathbb{Q}})$.

By rationalizing (6.46), we obtain a weak equivalence of commutative $KU_{\mathbb{Q}}$-algebras

$$THH^n(KU_{\mathbb{Q}}) \simeq KU_{\mathbb{Q}} \wedge K(\mathbb{Z}, a_1 + 2)_+ \wedge \cdots \wedge K(\mathbb{Z}, a_{2^n-1} + 2)_+.$$

Its homotopy is isomorphic to its rational homology, and rational homology satisfies a Künneth isomorphism. By using the identification of the rationalized Eilenberg-Mac Lane spaces of Section 4.3, we obtain

PROPOSITION 6.53. *There is an isomorphism of commutative* $\mathbb{Q}[t^{\pm 1}]$*-algebras*

$$THH^n_*(KU_{\mathbb{Q}}) \cong \mathbb{Q}[t^{\pm 1}] \otimes \bigotimes_{a_i \text{ odd}} E(\sigma^i t) \otimes \bigotimes_{a_j \text{ even}} \mathbb{Q}[\sigma^j t] \tag{6.54}$$

where $|\sigma^r t| = a_r + 2$ *and* $i, j \in \{1, \ldots, 2^n - 1\}$.

For example,

$$THH^2_*(KU_{\mathbb{Q}}) \cong \mathbb{Q}[t^{\pm 1}] \otimes E(\sigma t) \otimes E(\sigma t) \otimes \mathbb{Q}[\sigma^2 t]$$

with $|\sigma t| = 3$ and $|\sigma^2 t| = 4$. Note that, by splitting off the units of the latter three factors and distributing, this expression is coherent with (6.52), except we now understand the multiplicative structure.

We can recognize the expression (6.54) as an iterated Hochschild homology algebra:

$$THH^n_*(KU_{\mathbb{Q}}) \cong HH^{\mathbb{Q},n}_*(\mathbb{Q}[t^{\pm 1}]). \tag{6.55}$$

Indeed, $HH^{\mathbb{Q}}_*(\mathbb{Q}[t^{\pm 1}]) \cong \mathbb{Q}[t^{\pm 1}] \otimes E(\sigma t)$, and $HH^{\mathbb{Q}}_*(E(\sigma t)) \cong E(\sigma t) \otimes \mathbb{Q}[\sigma^2 t]$. These Hochschild homology calculations are classical and can be found e.g. in [**MS93**, Section 2] and [**AR05**, 2.4]. We use that localization commutes with Hochschild homology [**Wei94**, Theorem 9.1.8(3)]. Also note that in general, the Hochschild homology of an exterior algebra is isomorphic to the tensor product of this same exterior algebra with a divided power algebra, but over $\mathbb{Q}$ such algebras are polynomial.

We can also arrive at such an iterated Hochschild homology expression by a spectral sequence computation in rational homology. First, note that if A is a rational commutative $\mathbb{S}$-algebra, then there is a weak equivalence $THH(A) \xrightarrow{\sim} THH^{H\mathbb{Q}}(A)$. Indeed, this can be checked simplicially, the multiplication map $H\mathbb{Q} \wedge H\mathbb{Q} \to H\mathbb{Q}$ being a weak equivalence.

There are strongly convergent Bökstedt spectral sequences [**EKMM97**, IX.1.9], [**AR05**, 4.1]

$$E^2_{p,q}(n) = HH^{\mathbb{Q}}_{p,q}(H\mathbb{Q}_*(THH^{H\mathbb{Q},n-1}(KU_{\mathbb{Q}}))) \Rightarrow H\mathbb{Q}_{p+q}(THH^{H\mathbb{Q},n}(KU_{\mathbb{Q}}))$$

which we can express as

$$E^2_{p,q}(n) = HH^{\mathbb{Q}}_{p,q}(THH^{n-1}_*(KU_{\mathbb{Q}})) \Rightarrow THH^n_{p+q}(KU_{\mathbb{Q}}).$$

These are spectral sequences of commutative $\pi_*(KU_{\mathbb{Q}}) \cong \mathbb{Q}[t^{\pm 1}]$-algebras, and by induction on n they collapse, since the algebra generators are in filtration degree 0 and 1. Thus we obtain an isomorphism $E^2_{p,q}(n) \cong HH^{\mathbb{Q},n}_{p,q}(\mathbb{Q}[t^{\pm 1}])$.

Denote by $\overline{HH}^{\mathbb{Q},n}_*(B)$ the kernel of the augmentation $HH^{\mathbb{Q},n}_*(B) \to B$. In conclusion,

THEOREM 6.56. *There is an isomorphism of non-unital commutative $\mathbb{Q}[t^{\pm 1}]$-algebras*

$$\overline{THH}^n_*(KU) \cong \overline{HH}^{\mathbb{Q},n}_*(\mathbb{Q}[t^{\pm 1}]).$$

Of course, this is also the kernel of the augmentation of the right-hand side of (6.54) over $\mathbb{Q}[t^{\pm 1}]$, but alas, we do not see a slick notational device for it.

6. $\Sigma Y \otimes KU$

In this section, we evaluate the commutative KU-algebra $\Sigma Y \otimes KU$ when Y is a based CW-complex, by comparing it with $Y \otimes_{KU} (S^1 \otimes KU)$. We are very grateful to Bjørn Dundas for suggesting this line of argument.

Recall that if R is a commutative $\mathbb{S}$-algebra, the category R-**CAlg** is tensored over **Top** [**EKMM97**, VII.2.9]. If $A \in R$-**CAlg**, then the tensor $S^1 \otimes_R A$ is naturally isomorphic to $THH^R(A)$ as a commutative augmented A-algebra [**MSV97**], [**EKMM97**, IX.3.3], [**AR05**, Section 3]. Therefore, in this section we will identify $S^1 \otimes_R A$ and $THH^R(A)$ without further notice.

6.1. The morphism ν. Let $\mathcal{C}$ be a category enriched and tensored over **Top**. Denote its tensor by $\otimes$. Fix a pointed space (Z, z_0). We denote by ν^Z the natural transformation

$$\mathcal{C} \underset{Z\otimes -}{\overset{\mathrm{id}}{\Downarrow \nu^Z}} \mathcal{C} \tag{6.57}$$

whose component in $C \in \mathcal{C}$ is given by

$$\nu^Z_C \coloneqq \eta^C_Z(z_0) : C \to Z \otimes C. \tag{6.58}$$

Here $\eta_Z^C : Z \to \mathcal{C}(C, Z \otimes C)$ is the unit at Z of the adjunction

$$\mathbf{Top} \underset{\mathcal{C}(C,-)}{\overset{-\otimes C}{\rightleftarrows}} \mathcal{C}. \tag{6.59}$$

Let us now highlight the naturality properties of ν_C^Z at C and at Z. Let $\varphi : C \to C'$ be a morphism in $\mathcal{C}$. The naturality of the isomorphism

$$\mathcal{C}(Z \otimes C, Z \otimes -) \cong \mathbf{Top}(Z, \mathcal{C}(C, Z \otimes -))$$

gives the commutativity of the following diagram

$$\begin{array}{ccc} C & \xrightarrow{\nu_C^Z} & Z \otimes C \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi} \\ C' & \xrightarrow[\nu_{C'}^Z]{} & Z \otimes C' \end{array} \tag{6.60}$$

Let $u : Z \to Z'$ be a morphism of based spaces. The naturality of η^C gives the commutativity of the following diagram.

$$\begin{array}{ccc} C & \xrightarrow{\nu_C^Z} & Z \otimes C \\ & {\scriptstyle \nu_C^{Z'}} \searrow & \downarrow{\scriptstyle u\otimes\mathrm{id}} \\ & & Z' \otimes C \end{array} \tag{6.61}$$

Example 6.62. Let $\mathcal{C} = \mathbf{Top}_*$ be the category of pointed objects in $\mathbf{Top}$. It is tensored over $\mathbf{Top}$: if $X \in \mathbf{Top}_*$ and $Y \in \mathbf{Top}$, then $Y \otimes X$ is defined as $Y_+ \wedge X$. When (Y, y_0) is pointed, we denote by

$$n_X^Y : X \to Y_+ \wedge X \tag{6.63}$$

the map ν_X^Y of (6.58) applied to $\mathcal{C} = \mathbf{Top}_*$. More explicitely, the map n_X^Y takes X to the copy of X lying over y_0 in $Y_+ \wedge X$.

6.2. In commutative algebras. Let R be a commutative $\mathbb{S}$-algebra. Let A be a commutative R-algebra and (X, x_0) be a based space. The map (6.58) in this scenario is a map of commutative R-algebras

$$\nu_A^X : A \to X \otimes_R A$$

which gives $X \otimes_R A$ the structure of a commutative A-algebra. In particular, when $X = S^1$, this is the usual structure of an A-algebra of $THH^R(A)$.

Now, take $R = \mathbb{S}$ and $A = KU$. Let (Y, y_0) be a based space. We use the symbol $\otimes$ to denote the tensor of $\mathbb{S}$-$\mathbf{CAlg}$ over $\mathbf{Top}$. The following diagram in KU-$\mathbf{CAlg}$ commutes.

Here the map $e : S^1 \to *$ collapses the circle into its basepoint, and we have identified $F(* \wedge KU_{\mathbb{Q}})$ and $* \otimes KU$ with KU.

$$
\begin{array}{ccccc}
Y \otimes_{KU} F(S^1 \wedge KU_{\mathbb{Q}}) & \xleftarrow{\nu^Y_{F(S^1 \wedge KU_{\mathbb{Q}})}} & F(S^1 \wedge KU_{\mathbb{Q}}) & \xrightarrow{F(e \wedge \mathrm{id})} & KU \\
\sim \downarrow \mathrm{id} \otimes \tilde{f} & & \sim \downarrow \tilde{f} & & \downarrow \mathrm{id} \\
Y \otimes_{KU} (S^1 \otimes KU) & \xleftarrow[\nu^Y_{S^1 \otimes KU}]{} & S^1 \otimes KU & \xrightarrow{e \otimes \mathrm{id}} & KU
\end{array}
\tag{6.64}
$$

Indeed, the right square commutes because the $\tilde{f}$ of Theorem 6.27 is a morphism of augmented KU-algebras, and the commutativity of the left square is an application of the commutativity of (6.60). Note that $\mathrm{id} \otimes \tilde{f}$ is a weak equivalence because $Y \otimes_{KU} -$ is a left Quillen functor, assuming Y is a CW-complex.

We will now identify the members of the left column.

PROPOSITION 6.65. *Let (X, x_0) and (Y, y_0) be based spaces, and let A be a commutative R-algebra.*

(1) There is an isomorphism of commutative A-algebras

$$Y \otimes_A (X \otimes_R A) \cong (Y_+ \wedge X) \otimes_R A$$

*where $\otimes_R$ (resp. $\otimes_A$) denotes the tensoring of R-**CAlg** (resp. A-**CAlg**) over **Top**.*

*Moreover, the isomorphism makes the following diagram in A-**CAlg** commute. The morphism $n^Y_X : X \to Y_+ \wedge X$ was defined in (6.63).*

$$
\begin{array}{ccc}
X \otimes_R A & \xrightarrow{\nu^Y_{X \otimes_R A}} & Y \otimes_A (X \otimes_R A) \\
& \searrow^{n^Y_X \otimes \mathrm{id}} & \downarrow \cong \\
& & (Y_+ \wedge X) \otimes_R A
\end{array}
\tag{6.66}
$$

*(2) Let M be an A-module. Let $F : A$-**Mod** $\to A$-**CAlg** be the free commutative algebra functor. There is an isomorphism*

$$Y \otimes_A F(X \wedge M) \cong F(Y_+ \wedge X \wedge M)$$

making the following diagram commute.

$$
\begin{array}{ccc}
F(X \wedge M) & \xrightarrow{\nu^Y_{F(X \wedge M)}} & Y \otimes_A F(X \wedge M) \\
& \searrow^{F(n^Y_X \wedge \mathrm{id})} & \downarrow \cong \\
& & F(Y_+ \wedge X \wedge M)
\end{array}
$$

*In the expression $Z \wedge M$ for a based space Z we are using the standard tensoring of A-**Mod** over $\mathbf{Top}_*$, i.e. $Z \wedge M = \Sigma^\infty Z \wedge M$.*

PROOF. (1) Let B be a commutative A-algebra with unit $\varphi : A \to B$. Using the defining adjunction for $Y \otimes_A -$, we get a homeomorphism

$$A\text{-}\mathbf{CAlg}(Y \otimes_A (X \otimes_R A), B) \cong \mathbf{Top}(Y, A\text{-}\mathbf{CAlg}(X \otimes_R A, B)). \tag{6.67}$$

The morphisms of commutative A-algebras $X \otimes_R A \to B$ are the morphisms of commutative R-algebras $g : X \otimes_R A \to B$ making the following diagram commute:

$$\begin{array}{ccc} & A & \\ {\scriptstyle \nu_A^X} \swarrow & & \searrow {\scriptstyle \varphi} \\ X \otimes_R A & \xrightarrow[g]{} & B. \end{array}$$

Recalling the definition of ν, this means that

$$g \circ \eta_X^A(x_0) = \varphi. \tag{6.68}$$

The adjoint map of g by the defining adjunction of $- \otimes_R A$ is the map in **Top**

$$X \xrightarrow{\eta_X^A} R\text{-}\mathbf{CAlg}(A, X \otimes_R A) \xrightarrow{g_*} R\text{-}\mathbf{CAlg}(A, B). \tag{6.69}$$

Let the space R-$\mathbf{CAlg}(A, B)$ be pointed by $\varphi : A \to B$. The condition (6.68) on the map g is then translated to the adjoint (6.69) by stating that it is a pointed map, i.e. it takes x_0 to φ. Thus, continuing (6.67),

$$\mathbf{Top}(Y, A\text{-}\mathbf{CAlg}(X \otimes_R A, B)) \cong \mathbf{Top}(Y, U\mathbf{Top}_*(X, R\text{-}\mathbf{CAlg}(A, B))), \tag{6.70}$$

where $U : \mathbf{Top}_* \to \mathbf{Top}$ is the functor forgetting the basepoint. It is the right adjoint to the functor $(-)_+ : \mathbf{Top} \to \mathbf{Top}_*$ which adds a disjoint basepoint, so we continue:

$$\mathbf{Top}(Y, U\mathbf{Top}_*(X, R\text{-}\mathbf{CAlg}(A, B))) \cong U\mathbf{Top}_*(Y_+, \mathbf{Top}_*(X, R\text{-}\mathbf{CAlg}(A, B))).$$

Since $\mathbf{Top}_*(X, -) : \mathbf{Top}_* \to \mathbf{Top}_*$ is the right adjoint to $- \wedge X$, we get:

$$U\mathbf{Top}_*(Y_+, \mathbf{Top}_*(X, R\text{-}\mathbf{CAlg}(A, B))) \cong U\mathbf{Top}_*(Y_+ \wedge X, R\text{-}\mathbf{CAlg}(A, B)).$$

By the same argument proving (6.70), we get

$$U\mathbf{Top}_*(Y_+ \wedge X, R\text{-}\mathbf{CAlg}(A, B)) \cong A\text{-}\mathbf{CAlg}((Y_+ \wedge X) \otimes_R A, B).$$

In conclusion, we have a homeomorphism

$$A\text{-}\mathbf{CAlg}(Y \otimes_A (X \otimes_R A), B) \cong A\text{-}\mathbf{CAlg}((Y_+ \wedge X) \otimes_R A, B),$$

and the Yoneda lemma finishes the proof.

The isomorphism was established using a chain of adjunctions. Following this chain, one observes that both n_X^Y and $\nu_{X \otimes_R A}^Y$, which are defined via units of adjunctions by analogous procedures, make the diagram (6.66) commute.

(2) The functor F is defined via a *continuous* monad in A-**Mod** (i.e. it is enriched over **Top**), see [**EKMM97**, proof of VII.2.9]. Therefore, the functor F is the left adjoint of a **Top**-adjunction [**Lin69**, Theorem 1]. In particular, F preserves tensors over **Top** [**Rie14**, 3.7.10], so we get the desired isomorphism. $\square$

Applying the previous proposition to $R = \mathbb{S}$, $A = KU$, $X = S^1$ and $M = KU_{\mathbb{Q}}$, the diagram (6.64) can be replaced with the following one.

(6.71)
$$\begin{array}{ccccc}
F(Y_+ \wedge S^1 \wedge KU_{\mathbb{Q}}) & \xleftarrow{F(n^Y_{S^1}\wedge \mathrm{id})} & F(S^1 \wedge KU_{\mathbb{Q}}) & \xrightarrow{F(e\wedge \mathrm{id})} & KU \\
\downarrow{\sim} & & \sim\downarrow{\tilde{f}} & & \downarrow{\mathrm{id}} \\
(Y_+ \wedge S^1) \otimes KU & \xleftarrow{n^Y_{S^1}\otimes \mathrm{id}} & S^1 \otimes KU & \xrightarrow{e\otimes \mathrm{id}} & KU
\end{array}$$

When Y is a based CW-complex, the left map is a weak equivalence.

Now, note that the following is a pushout square of based or unbased spaces.

$$\begin{array}{ccc}
S^1 & \xrightarrow{e} & * \\
{\scriptstyle n^Y_{S^1}}\downarrow & & \downarrow \\
Y_+ \wedge S^1 & \longrightarrow & Y \wedge S^1
\end{array}$$

Since the functors $- \otimes KU : \mathbf{Top} \to KU\text{-}\mathbf{CAlg}$ and $F(- \wedge KU_{\mathbb{Q}}) : \mathbf{Top}_* \to KU\text{-}\mathbf{CAlg}$ are left adjoints, they preserve pushouts, hence we get an induced map

$$\tau_Y : F(Y \wedge S^1 \wedge KU_{\mathbb{Q}}) \to (Y \wedge S^1) \otimes KU.$$

This is the component in Y of a natural transformation

(6.72)
$$\mathbf{Top}_* \underset{(-\wedge S^1)\otimes KU}{\overset{F(-\wedge S^1 \wedge KU_{\mathbb{Q}})}{\rightrightarrows}} KU\text{-}\mathbf{CAlg} \qquad \Downarrow \tau$$

as follows from the naturality of $n^Y_{S^1}$ in Y (6.61).

Suppose Y is a CW-complex. The three vertical maps of (6.71) are weak equivalences. The horizontal maps pointing left are cofibrations: indeed, $n^Y_{S^1}$ is a cofibration, $KU_{\mathbb{Q}}$ is a cofibrant KU-module (similarly as in Remark 6.34) so $KU_{\mathbb{Q}} \wedge -$ is left Quillen, F is left Quillen and $- \otimes KU$ is left Quillen. Moreover, all the objects are cofibrant in KU-**CAlg**. Therefore, as in any model category, the induced map of pushouts τ_Y is a weak equivalence. This proves the following theorem.

THEOREM 6.73. *There is a weak equivalence of commutative KU-algebras*

$$\tau_Y : F(Y \wedge S^1 \wedge KU_{\mathbb{Q}}) \to (Y \wedge S^1) \otimes KU$$

natural in the based CW-complex Y.

This determines $\Sigma Y \otimes KU$ as the free commutative KU-algebra on the KU-module $\Sigma Y \wedge KU_{\mathbb{Q}}$, up to weak equivalence. In particular, we have a weak equivalence of commutative KU-algebras

$$F(\Sigma^n KU_{\mathbb{Q}}) \to S^n \otimes KU \tag{6.74}$$

for every $n \geq 1$.

As in Remark 6.34, the KU-modules $\Sigma^n KU_{\mathbb{Q}}$ are cofibrant for $n \geq 0$. Since F is a left Quillen functor, Bott periodicity implies that we have weak equivalences

$$S^n \otimes KU \xleftarrow{\sim} \begin{cases} F(\Sigma KU_{\mathbb{Q}}) & \text{if } n \text{ is odd,} \\ F(KU_{\mathbb{Q}}) & \text{if } n \text{ is even} \end{cases} \tag{6.75}$$

for every $n \geq 1$.

7. A remark about MUP

In the same papers where Snaith gave the description of KU used in Section 4 [**Sna79**], [**Sna81**], he proved that the periodic complex cobordism ring spectrum, $MUP \simeq \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} MU$, can be constructed as $\mathbb{S}[BU][x^{-1}]$ for a generator $x \in \pi_2(\mathbb{S}[BU])$.

We would like to say that Theorem 6.18 gives an equivalence of commutative MUP-algebras

$$THH(MUP) \simeq MUP[SU], \tag{6.76}$$

since $BBU \simeq SU$ by Bott periodicity. However, BU is not a topological abelian group, but rather a group-like E_∞-space, so Theorem 6.18 does not really apply as is. However, we believe the result should hold. An approach to computing $THH(MUP)$ by considering MUP as a graded Thom spectrum can be found in [**SS14**].

Similarly, we would also like to say that Theorem 6.43 gives a weak equivalence of commutative MUP-algebras

$$THH^n(MUP) \simeq MUP\left[\prod_{i=1}^{n} (B^i SU)^{\times\binom{n}{i}}\right], \tag{6.77}$$

so e.g. for $n = 2$ we would get $THH^2(MUP) \simeq MUP[BSU \times BSU \times B^2SU]$.

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