THH and the monadic bar construction

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Contents

In this note, **Top** will denote the category of weakly Hausdorff compactly generated spaces and its objects will be called *spaces*.

We will work with the theory of [1]: we work with commutative S-algebras R, R-algebras A and A-modules.

1 Dold-Kan correspondence

We briefly recall some results on the Dold-Kan correspondence. Let $\mathcal{A}$ be an abelian category.

We denote by $s\mathcal{A}$ the category of simplicial objects in $\mathcal{A}$ and by $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$ which are zero in negative degrees. We define two functors $C, N : s\mathcal{A} \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$.

If $C_\bullet$ is a simplicial object in $\mathcal{A}$ with face maps d_i, define the *Moore complex of* $C_\bullet$ as

$$CC_\bullet = \left(C_n, \sum_{i=0}^{n} (-1)^i d_i \right)_n . \tag{1}$$

We define the *normalized complex* of $C_\bullet$ which we denote by $NC_\bullet$ by

$$NC_n = \bigcap_{i=0}^{n-1} \ker(d_i : C_n \to C_{n-1})$$

with differential $(-1)^n d_n$. It is a subcomplex of $C_\bullet$.

Theorem 1.1 (Dold-Kan). *[7, (8.4.1)] The functor $N : s\mathcal{A} \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$ is an equivalence of categories.*

The *homotopy groups* of $C_\bullet \in s\mathcal{A}$ are defined as

$$\pi_*(C_\bullet) = H_*(N(C_\bullet)).$$

Proposition 1.2. *[3, (2.2.4)] If $C_\bullet \in s\mathcal{A}$, the inclusion $i : NC_\bullet \to CC_\bullet$ is a natural chain homotopy equivalence. Thus we get a natural isomorphism $\pi_*(C_\bullet) \cong H_*(CC_\bullet)$.*

Let k be a commutative ring. If $C_\bullet$ is a simplicial k-module, we denote by $|C_\bullet|$ the geometric realization of the underlying simplicial set.

Proposition 1.3. *Let $C_\bullet$ be a simplicial k-module. Then we have a natural isomorphism*

$$\pi_*(|C_\bullet|) \cong \pi_*(C_\bullet).$$

This follows from the description of $\pi_n(C_\bullet)$ as $[\Delta^n/\partial\Delta^n, C_\bullet]$ and the fact that the geometric realization functor from the category of simplicial sets to the category of spaces is a Quillen equivalence. Indeed, $|\Delta^n/\partial\Delta^n| = S^n$. See [2, example 2.15].

Combining the previous two propostions we get a natural isomorphism in $C_\bullet \in s\mathcal{A}$:

$$\pi_*(|C_\bullet|) \cong H_*(CC_\bullet).$$

2 The bar construction

In this section we consider the classical algebraic bar constructions, all of which fall under the umbrella of the two-sided bar construction. These are important because they yield canonical resolutions, or as Cartan-Eilenberg called them, standard complexes.

Let k be a commutative ring and A be a (unital, associative) k-algebra; we denote $\otimes = \otimes_k$. Let M be a right A-module and N be a left A-module. We define a simplicial k-module $B_\bullet(M, A, N)$ called the *two-sided bar construction associated to A, M and N*, as follows:

$$B_n(M, A, N) = M \otimes A^{\otimes n} \otimes N,$$

with face maps

$$d_i : B_n(M, A, N) \to B_{n-1}(M, A, N)$$

and degeneracy maps

$$s_i : B_n(M, A, N) \to B_{n+1}(M, A, N)$$

given by

$$d_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \quad \text{and} \tag{2}$$
$$s_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for $i = 0, \dots, n$, where $a_0 \in M$, $a_1, \dots, a_n \in A$, $a_{n+1} \in N$. The simplicial identities are readily checked. We denote its associated Moore complex $CB_\bullet(M, A, N)$ by $B_*(M, A, N)$.

Let $A^e := A \otimes A^{\mathrm{op}}$ be the enveloping algebra of A. Then the category of (A, A)-bimodules is equivalent to both the category of left A^e-modules and the category of right A^e-modules. Explicitely, an (A, A)-bimodule M begets a left A^e-module M by the formula $(a \otimes a')m = ama'$, and a right A^e-module M by the formula $m(a \otimes a') = a'ma$. This holds in particular for $M = A$.

Proposition 2.1. 1. *The chain complex $B_*(A, A, A)$ yields a resolution of A as a left A^e-module with augmentation given by the multiplcation.*

If A is flat as a k-module, then this is a flat resolution.

2. *If N is a left A-module, then $B_*(A, A, N)$ yields a resolution of N as a left A-module, with augmentation given by the action of A on N.*

   If A and N are k-flat, then this is a flat resolution.

3. *If M is a flat right A-module, then $B_*(M, A, N)$ yields a resolution of $M \otimes_A N$ as a k-module, with augmentation given by the obvious map $M \otimes N \to M \otimes_A N$.*

   If moreover A and N are k-flat, then this is a flat resolution.

Proof. 1. Explicitely, $B_*(A, A, A)$ is the following chain complex of k-modules:

$$B_*(A,A,A): \qquad \dots \xrightarrow{d} A^{\otimes n+2} \xrightarrow{d} A^{\otimes n+1} \xrightarrow{d} \dots \xrightarrow{d} A^{\otimes 2} \longrightarrow 0. \tag{3}$$

Here $A^{\otimes n+2}$ is in degree n, and $d : A^{\otimes n+2} \to A^{\otimes n+1}$ is given by $d = \sum_{i=0}^{n} (-1)^i d_i$.

This is actually a complex of (A, A)-bimodules. Moreover, if A is k-flat then $A^{\otimes n+2}$ is so too as an A^e-module, for all $n \geq 0$. Indeed, $A^{\otimes n+2} = A \otimes A^{\otimes n} \otimes A \cong A \otimes A^{\mathrm{op}} \otimes A^{\otimes n}$ is the extension of scalars of the flat k-module $A^{\otimes n}$ to A^e, so it is still flat.

Let us augment the complex (3) by $\mu : A^{\otimes 2} \to A$, the multiplication of A:

$$\dots \xrightarrow{d} A^{\otimes n+2} \xrightarrow{d} A^{\otimes n+1} \xrightarrow{d} \dots \xrightarrow{d} A^{\otimes 2} \xrightarrow{\mu} A \longrightarrow 0. \tag{4}$$

This is still a complex of (A, A)-bimodules. To prove it is exact, we build a contracting homotopy. Let $s : A^{\otimes n+1} \to A^{\otimes n+2}$ be an "extra degeneracy":

$$s(a_0 \otimes \dots \otimes a_n) = 1 \otimes a_0 \otimes \dots \otimes a_n.$$

It is quickly verified that $d_i s = s d_{i-1}$ for all $i > 1$ and $d_0 s = \mathrm{id}$. So $ds + sd = \mathrm{id}$ and also $ds + s\mu = \mathrm{id}$, so s is a contracting homotopy of the complex (4).

2. Apply the functor $- \otimes_A N$ to the exact complex (4). It is now a complex of left A-modules, and it is easily identified with $B_*(A, A, N)$ augmented by the action of A on N. The contracting homotopy we defined carries over, mutatis mutandis.

   If A is k-flat then the bar resolution $B(A, A, A)$ is flat; upon tensoring it with the flat k-module N it remains flat as an A-module.

3. If M is a right flat A-module, the functor $M \otimes_A -$ is exact and lands in k-modules. Apply it to what we obtained in 2), and again it is easy to recognize it yields what we want. Observe that we cannot define the contracting homotopy when having M as a first factor, hence the hypothesis of flatness is necessary for obtaining exactness. □

Remark 2.2. Let us record two isomorphisms found on the way: we have an isomorphism of left A-module complexes

$$B_*(A, A, N) \cong B_*(A, A, A) \otimes_A N$$

and an isomorphism of k-module complexes

$$B_*(M, A, N) \cong M \otimes_A B(A, A, N) = (M \otimes N) \otimes_{A^e} B_*(A, A, A).$$

3 Hochschild homology

Let k be a commutative ring and A be a (unital, associative) k-algebra; we denote $\otimes = \otimes_k$. Let M be an (A, A)-bimodule.

3.1 As (the homotopy groups of) a simplicial module

We define a simplicial k-module $\mathcal{H}_\bullet(A, M)$ as follows:

$$\mathcal{H}_n(A, M) = M \otimes A^{\otimes n},$$

with face maps $d_i : \mathcal{H}_n(A, M) \to \mathcal{H}_{n-1}(A, M)$ and degeneracy maps $s_i : \mathcal{H}_n(A, M) \to \mathcal{H}_{n+1}(A, M)$ given by

$$d_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{if } i = 0, \ldots, n-1 \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n, \end{cases}$$

$$s_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \qquad \text{if } i = 0, \ldots, n,$$

where $a_0 \in M$ and $a_1, \ldots, a_n \in A$. The simplicial identities are readily checked.

We define the *Hochschild homology of A with coefficients in M*:

$$\mathcal{H}_*(A, M) \coloneqq \pi_*(\mathcal{H}_\bullet(A, M)).$$

When we take $M = A$ we denote $\mathcal{HH}_*(A) \coloneqq \mathcal{H}_*(A, A)$ and we call it the *Hochschild homology of A*.

3.1.1 As (the homotopy groups of) a space

By proposition 1.3, we get that

$$\mathcal{H}_*(A, M) = \pi_*(|\mathcal{H}_\bullet(A, M)|).$$

3.1.2 As (the homology groups of) a chain complex

We can consider the associated Moore complex $C\mathcal{H}_\bullet(A, M)$ to the simplicial module $\mathcal{H}(A, M)$ as in (1). We call it the *Hochschild complex* of A with coefficients in M.

By proposition 1.2, we get that

$$\mathcal{H}_*(A, M) = H_*(C\mathcal{H}_\bullet(A, M)).$$

3.2 As a derived tensor product

We can give yet another expression for $\mathcal{H}_*(A, M)$, if we accept an additional flatness hypothesis.

Proposition 3.1. *Suppose A is flat as a k-module. Then*

$$\mathcal{H}_*(A, M) = \mathrm{Tor}_*^{A^e}(M, A). \tag{5}$$

Proof. Consider $B(A, A, A)$, the bar resolution of A as in proposition 2.1. It is a flat resolution of A as an A^e-module. Let us apply $M \otimes_{A^e} -$ to (3): we obtain a complex of A^e-modules,

$$\ldots \xrightarrow{b'} M \otimes_{A^e} A^{\otimes n+2} \xrightarrow{b'} M \otimes_{A^e} A^{\otimes n+1} \xrightarrow{b'} \ldots \xrightarrow{b'} M \otimes_{A^e} A^{\otimes 2} \xrightarrow{\mu} M \otimes_{A^e} A \longrightarrow 0. \tag{6}$$

Since Tor can be computed with flat resolutions, the homology of this complex computes the Tor in (5). I claim that this complex is isomorphic to the Hochschild complex of A with coefficients in M, and so we obtain the result.

Indeed, first observe that there is an isomorphism of A^e-modules $M \otimes_{A^e} A^{\otimes n+2} \cong M \otimes A^{\otimes n}$:

$$M \otimes_{A^e} \otimes A^{\otimes n+2} = M \otimes_{A^e} A \otimes A^{\otimes n} \otimes A \cong M \otimes_{A^e} \otimes A^e \otimes A^{\otimes n} \cong M \otimes A^{\otimes n}.$$

An inspection of the differentials shows that the following ladder diagram commutes, and hence that we have the desired isomorphism of complexes of A^e-modules.

$$\begin{tikzcd}
M \otimes_{A^e} B_*(A,A,A): \arrow[d, "\cong"'] & \cdots \arrow[r] & M \otimes_{A^e} A^{\otimes n+2} \arrow[r] \arrow[d, "\cong"'] & M \otimes_{A^e} A^{\otimes n+1} \arrow[r] \arrow[d, "\cong"'] & \cdots \arrow[r] & M \otimes_{A^e} A^{\otimes 2} \arrow[r] \arrow[d, "\cong"'] & 0 \\
C\mathcal{H}_*(A,M): & \cdots \arrow[r] & M \otimes A^{\otimes n} \arrow[r] & M \otimes A^{\otimes n-1} \arrow[r] & \cdots \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

□

Remark 3.2.
1. From the end of the previous proof we deduce that we can recover the bar complex $B(A,A,A)$ as a Hochschild complex: $B_*(A,A,A) \cong C\mathcal{H}_\bullet(A,A^e)$.
2. The Hochschild chain complex $C\mathcal{H}_\bullet(A,A) \cong M \otimes_{A^e} B_*(A,A,A)$ is called the *cyclic bar construction* associated to A. This is reasonable since it is an example of a cyclic k-module (see [4]).

With enough flatness over k the bar resolution gives us a canonical way to compute Tor, and in fact identifies it with Hochschild homology modules with coefficients in the tensor product:

Corollary 3.3. *Let M be a right A-module and N be a left A-module. Suppose A and N are flat as k-modules. Then we have an isomorphism of k-modules*

$$\mathcal{H}_*(A, M \otimes N) = \mathrm{Tor}_*^A(M,N).$$

Proof. Consider the bar resolution $B_*(A,A,A)$. It gives a flat resolution of A as an A^e-module. If we apply $(M \otimes N) \otimes_{A^e} -$ to it, the homology of the resulting complex computes $\mathrm{Tor}_*^{A^e}(M \otimes N, A) \cong \mathcal{H}_*(A, M \otimes N)$ by proposition 3.1.

On the other hand, $(M \otimes N) \otimes_{A^e} - \cong (M \otimes_A -) \circ (- \otimes_A N)$. If we first apply $- \otimes_A N$ to $B_*(A,A,A)$, we get $B_*(A,A,N)$ (remark 5.4). In view of proposition 2.1.2, this yields a flat resolution of N as a left A-module. So if we now apply $M \otimes_A -$ to it and compute its homology, we get $\mathrm{Tor}_*^A(M,N)$. This finishes the proof. □

4 The geometric realization of simplicial spectra

4.1 Coends

We recall the definition of a special kind of colimit. Let I be a small category and $\mathcal{C}$ be a cocomplete category. Let $F : I^{op} \times I \to \mathcal{C}$ be a functor. Then the *coend* of F is the following coequalizer:

$$\int^{n \in I} F(n,n) := \mathrm{colim}\left(\coprod_{f: i \to j} F(j,i) \rightrightarrows \coprod_k F(k,k) \right)$$

where the two arrows are the following. If $f : i \to j$ is an arrow in I, we have two arrows given as the following compositions: $F(j,i) \xrightarrow{F(f^{op}, \mathrm{id}_i)} F(i,i) \hookrightarrow \coprod_k F(k,k)$ and $F(j,i) \xrightarrow{F(\mathrm{id}_j, f)} F(j,j) \hookrightarrow \coprod_k F(k,k)$. This defines two maps as wanted.

4.2 A commentary on the general framework

A general framework in which one can define the geometric realization of a simplicial object is for categories $\mathcal{C}$ which are enriched over **sSet**, tensored[1] and cocomplete. In this setting, if $X_\bullet \in s\mathcal{C}$, one defines ([6,

[1] Also called *copowered*. A category $\mathcal{C}$ enriched over a monoidal category $\mathcal{V}$ is *tensored* if for every $V \in \mathcal{V}$ and $C \in \mathcal{C}$ there exists an object $V \otimes C \in \mathcal{C}$ such that there is an isomorphism in $\mathcal{V}$, $\mathcal{C}(V \otimes C, C') \cong \mathcal{V}(V, \mathcal{C}(C, C'))$.

(3.8.1)])
$$|X_\bullet| = \int^{[n]\in\Delta} \Delta^n \otimes X_n \in \mathcal{C}$$
where $\Delta^n \in \mathbf{sSet}$ is the standard n-simplex, so that geometric realization is a functor $|-| : s\mathcal{C} \to \mathcal{C}$.

Somewhat dually, one might define it for categories enriched over $\mathbf{Top}$, as
$$|X_\bullet| = \int^{[n]\in\Delta} \Delta_n \otimes X_n \in \mathcal{C} \tag{7}$$
where $\Delta_n \in \mathbf{Top}$ is the standard topological n-simplex.

Some classical categories $\mathcal{C}$ that satisfy either of the hypotheses above are simplicial sets and spaces. The enrichment for the category of spaces over $\mathbf{sSet}$ and of the category of simplicial sets over $\mathbf{Top}$ come from the adjunction of classical geometric realization and singular simplicial set. See [6, (6.2.2)]. As is expected, in these cases any two of the definitions above yield the same object.

For the above paragraph not to be circular, we are assuming that classical geometric realization has already been defined. This is not troubling, because classical geometric realization $\mathbf{sSet} \to \mathbf{Top}$ is not encompassed by the definitions above (the codomain should be $\mathbf{Set}$). In any case, classical geometric realization is defined in a very similar way to (7), but taking an additional step: we take X_n to be a discrete topological space and we compute the product and the colimit in $\mathbf{Top}$.

Remark 4.1. Let $\mathcal{C}$ be a category as above. Let $C \in \mathcal{C}$ and consider the constant simplicial object $C_\bullet$, defined as $C_n = C$ for all n and all maps between simplices are defined to be identities. Then $|C_\bullet| = C$. This follows from general categorical arguments.

4.3 For simplicial spectra

Let R be a commutative S-algebra and $K_\bullet$ be a simplicial R-module. We define (or we get from the previous section) its geometric realization $|K_\bullet|$, which is an R-module:
$$|K_\bullet| = \int^{[n]\in\Delta} K_n \wedge (\Delta_n)_+.$$
Here $(\Delta_n)_+$ is the standard topological n-simplex with a point attached. Geometric realization defines a functor from the category of simplicial R-modules to the category of R-modules.

Proposition 4.2. *[1, (X.1.2)] The geometric realization functor preserves homotopies.*

In the previous proposition, "homotopies" in the category of simplicial R-modules can be understood in two ways. It can mean simplicial maps with domains of the form $K_\bullet \wedge I_+$, or it can mean the combinatorial kind of simplicial homotopy that makes sense for any category [5, (9.1)]. Both are preserved.

Proposition 4.3. *[1, (X.1.3)] For simplicial R-modules $K_\bullet$, $L_\bullet$ and simplicial based spaces $X_\bullet$, there are natural isomorphisms:*

1. $|K_\bullet \wedge L_\bullet| \cong |K_\bullet| \wedge |L_\bullet|$,
2. $|K_\bullet \wedge X_\bullet| \cong |K_\bullet| \wedge |X_\bullet|$,
3. $\Sigma^\infty |X_\bullet| \cong |\Sigma^\infty X_\bullet|$.

There is a technical condition one can impose on a simplicial R-module: we can ask for it to be *proper*, see definitions [1, (X.2.1), (X.2.2)].

Proposition 4.4. *Let $f_\bullet : K_\bullet \to L_\bullet$ be a map of proper simplicial R-modules. If $f_n : K_n \to L_n$ is a homotopy equivalence (resp. weak equivalence) for all n, then $|f_\bullet| : |K_\bullet| \to |L_\bullet|$ is a homotopy equivalence (resp. weak equivalence).*

5 The bar construction for monads

5.1 Monads

Definition 5.1. A *monad* on a category $\mathcal{C}$ is a triple (T, η, μ) where $T : \mathcal{C} \to \mathcal{C}$ is a functor and $\eta : \mathrm{id}_{\mathcal{C}} \to T$, $\mu : T^2 \to T$ are natural transformations such that the following diagrams commute.

$$
\begin{tikzcd}
T^2 \arrow[dr, "\mu"'] & T \arrow[l, "T\eta"'] \arrow[r, "\eta T"] \arrow[d, "\mathrm{id}_T"] & T^2 \arrow[dl, "\mu"] \\
& T &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
$$

The natural transformations η and μ are called the *unit* and the *multiplication* of the monad.

If $\mathcal{C}$ is small (or if we sweep size issues under the carpet), we can reformulate the definition as: a monad is a monoid object in the category of endofunctors $\mathcal{C}^{\mathcal{C}}$.

The basic machine for producing examples is the following. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be functors and $\eta : \mathrm{id}_{\mathcal{C}} \to GF$, $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ be natural transformations such that (F, G, η, ϵ) is an adjunction (i.e. F is left adjoint to G with unit η and counit ϵ). Then $(GF, \eta, G\epsilon F)$ is a monad on $\mathcal{C}$.

Definition 5.2. If (T, η, μ) is a monad on a category $\mathcal{C}$, a *T-algebra* is a pair (C, α) where $C \in \mathcal{C}$ and $\alpha : TC \to C$ is an arrow in $\mathcal{C}$, called the *structure map*, such that the following diagrams commute.

$$
\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "\mathrm{id}_C"'] & TC \arrow[d, "\alpha"] \\
& C
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2C \arrow[r, "\mu_C"] \arrow[d, "T\alpha"'] & TC \arrow[d, "\alpha"] \\
TC \arrow[r, "\alpha"'] & C
\end{tikzcd}
$$

If (C, α) and (C', α') are two T-algebras, then a map $(C, \alpha) \to (C', \alpha')$ is given by a map $f : C \to C'$ in $\mathcal{C}$ such that the following diagram commutes.

$$
\begin{tikzcd}
TC \arrow[r, "\alpha"] \arrow[d, "Tf"'] & C \arrow[d, "f"] \\
TC' \arrow[r, "\alpha'"'] & C'
\end{tikzcd}
$$

Thus we get a category of T-algebras.

Definition 5.3. [1, (II.6.3)] Let (T, η, μ) be a monad in a category $\mathcal{C}$. A *right T-functor* in a category $\mathcal{C}'$ is a pair (F, ν) where $F : \mathcal{C} \to \mathcal{C}'$ is a functor and $\nu : FT \to F$ is a natural transformation called *right action* such that the following diagrams commute.

$$
\begin{tikzcd}
FT \arrow[dr, "\nu"'] & F \arrow[l, "F\eta"'] \arrow[d, "\mathrm{id}_F"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
FT^2 \arrow[r, "F\mu"] \arrow[d, "\nu T"'] & FT \arrow[d, "\nu"] \\
FT \arrow[r, "\nu"'] & F
\end{tikzcd}
$$

Let (T', η', μ') be a monad in a category $\mathcal{C}'$. A *left T'-functor* in a category $\mathcal{C}$ is a pair (F, λ) where $F : \mathcal{C} \to \mathcal{C}'$ is a functor and $\lambda : T'F \to F$ is a natural transformation called *left action* such that the following diagrams commute.

$$
\begin{tikzcd}
F \arrow[r, "\eta' F"] \arrow[d, "\mathrm{id}_F"'] & T'F \arrow[dl, "\lambda"] \\
F &
\end{tikzcd}
\qquad
\begin{tikzcd}
T'^2F \arrow[r, "\mu' F"] \arrow[d, "T'\lambda"'] & T'F \arrow[d, "\lambda"] \\
T'F \arrow[r, "\lambda"'] & F
\end{tikzcd}
$$

A *(T',T)-functor* is a triple (F,λ,ν) where (F,λ) is a left T'-functor, (F,ν) is a right T-functor and such that the following diagram commutes.

$$\begin{tikzcd} T'FT \arrow[r, "\lambda T"] \arrow[d, "T'\nu"'] & FT \arrow[d, "\nu"] \\ T'F \arrow[r, "\lambda"'] & F \end{tikzcd}$$

Remark 5.4. Let (T',η',μ') be a monad in a category $\mathcal{C}'$. If $(F:\mathcal{C}\to\mathcal{C}',\lambda)$ is a left T'-functor and $C\in\mathcal{C}$, then (FC,λ_C) is a T'-algebra.

Example 5.5. Let $\mathcal{C}$ be a category with a monad (T,η,μ). Then $T:\mathcal{C}\to\mathcal{C}$ is a (T,T)-functor, taking ν and λ to be μ.

5.2 The bar construction

Definition 5.6. [1, (XII.1.1)] Let (T,η,μ) be a monad in a category $\mathcal{C}$, (C,α) be a T-algebra and (F,ν) be a right T-functor on a category $\mathcal{C}'$. We define a simplicial object $B_\bullet(F,T,C)$ in $\mathcal{C}'$ as follows: $B_n(F,T,C)=FT^nC$ with faces and degeneracies given by

$$d_i=\begin{cases}(\nu T^{n-1})_C & \text{if } i=0\\ (FT^{i-1}\mu T^{n-i-1})_C & \text{if } 0<i<n\\ FT^{n-1}\alpha & \text{if } i=n\end{cases}$$

and $s_i=(FT^i\eta T^{n-i})_C$.

Remark 5.7. If $\mathcal{C}'$ has a monad (T',η',μ') and F is a (T',T)-functor then $B_\bullet(F,T,C)$ is a simplicial T'-algebra, i.e. it is a simplicial object in the category of T'-algebras, where $B_n(F,T,C)=FT^nC$ is a T'-algebra with the structure map α given by $\lambda_{T^nC}:T'FT^nC\to FT^nC$, where $\lambda:T'F\to F$ is the left action of F (cf. remark 5.4).

Example 5.8. Following example 5.5, if T is a monad in a category $\mathcal{C}$ and C is a T-algebra, we can form the bar construction $B_\bullet(T,T,C)$ and it is a simplicial T-algebra by the previous remark.

The following proposition exhibits $B_\bullet(T,T,C)$ as a "simplicial resolution" of C.

Proposition 5.9. *Let (T,η,μ) be a monad in a category $\mathcal{C}$ and (C,α) be a T-algebra. There is a natural simplicial homotopy equivalence*

$$B_\bullet(T,T,C)\to C_\bullet$$

in the category $s\mathcal{C}$, where $C_\bullet$ is the constant simplicial T-algebra associated to C, i.e. $C_n=C$ for all n and all faces and degeneracies are taken to be identities.

Sketch of proof: Define $\varepsilon_\bullet:B_\bullet(T,T,C)\to C_\bullet$ and $\varphi_\bullet:C_\bullet\to B_\bullet(T,T,C)$ with $\varepsilon_n:T^{n+1}C\to C$ and $\varphi_n:C\to T^{n+1}C$ given by

$$\varepsilon_n=\begin{cases}\alpha & \text{if } n=0,\\ \mu_C\circ\alpha & \text{if } n=1,\\ \mu_{T^{n-1}C}\circ\cdots\circ\mu_{TC}\circ\mu_C\circ\alpha & \text{if } n\geq 2.\end{cases}\qquad \varphi_n=\begin{cases}\eta_C & \text{if } n=0,\\ \eta_C\circ\eta_{TC}\circ\cdots\circ\eta_{T^nC} & \text{if } n\geq 1.\end{cases}$$

Then $\varepsilon_\bullet\circ\varphi_\bullet=\mathrm{id}$, and there is a simplicial homotopy $\varphi_\bullet\circ\varepsilon_\bullet\simeq\mathrm{id}$ ([5, (9.8)]). $\square$

Functoriality Let (T,η,μ) be a monad in a category $\mathcal{C}$, let (C,α) be a T-algebra and let (F,ν), (F',ν') be two right T-functors. A *morphism* $(F,\nu)\to(F',\nu')$ is given by a natural transformation $\pi : F\to F'$ such that the following diagram commutes.

$$\begin{tikzcd} FT \arrow[r, "\nu"] \arrow[d, "\pi T"'] & F \arrow[d, "\pi"] \\ F'T \arrow[r, "\nu'"'] & F' \end{tikzcd}$$

Such a morphism induces a simplicial map between bar constructions:

$$B_\bullet(F,T,C)\to B_\bullet(F',T,C)$$

defined as $\pi_{T^nC} : FT^nC\to F'T^nC$.

This gives functoriality of the bar construction in the first variable. It is actually functorial in all three of them but we don't need it for our purposes.

5.3 The case of modules over a ring

The monad Let k be a commutative ring, A be a k-algebra. Extension and restriction of scalars with respect to the unit map $k\to A$ form an adjoint pair of functors.

$$\begin{tikzcd} \mathbf{k\text{-}Mod} \arrow[r, bend left, "A\otimes -"] & \mathbf{A\text{-}Mod} \arrow[l, bend left, "U"] \end{tikzcd}$$

The associated monad (T,η,μ) is such that, for $M\in\mathbf{k\text{-}Mod}$:

$$\begin{aligned} T : \mathbf{k\text{-}Mod}\to\mathbf{k\text{-}Mod}, &\qquad M\mapsto A\otimes M \\ \eta_M : M\to A\otimes M, &\qquad m\mapsto 1\otimes m \\ \mu_M : A\otimes(A\otimes M)\to A\otimes M, &\qquad a'\otimes(a\otimes m)\mapsto a'a\otimes m. \end{aligned}$$

The algebra An algebra over this monad is a left A-module N with structure map $A\otimes N\to N$ given by the action of A on N.

The functor Let M be a right A-module. Let F be the functor $M\otimes - : \mathbf{k\text{-}Mod}\to\mathbf{k\text{-}Mod}$, and let $\nu : FT\to T$ be the natural transformation given by $\nu_P : M\otimes A\otimes P\to M\otimes P$, $m\otimes a\otimes p\mapsto ma\otimes p$. Thus (F,ν) is a right T-functor.

If M is an (A,A)-bimodule, then this functor is a (T,T)-functor, where the left action $\lambda_P : TFP = A\otimes M\otimes P\to M\otimes P = FP$ is given by $a\otimes m\otimes p\mapsto am\otimes p$.

And the bar construction. It is now clear that the classical two-sided bar construction of section 2 is a special case of the monadic one, only by expressing the faces and degeneracies (2) of the classical simplicial k-module $B_\bullet(M,A,N)$ in an element-free way. Let $\mu : A\otimes A\to A$ be the product of A and $\eta : k\to A$ be its unit map. Let $\nu : M\otimes A\to A$ and $\alpha : A\otimes N\to N$ be the actions of A on M and N. Then we have

$$d_i=\begin{cases}\nu\otimes\mathrm{id}_A^{\otimes n-1}\otimes\mathrm{id}_N & \text{if } i=0,\\ \mathrm{id}_M\otimes\mathrm{id}_A^{\otimes i-1}\otimes\mu\otimes\mathrm{id}_A^{\otimes n-i-1}\otimes\mathrm{id}_N & \text{if } 0<i<n,\\ \mathrm{id}_M\otimes\mathrm{id}_A^{\otimes n-1}\otimes\alpha & \text{if } i=n\end{cases}$$

and $s_i=\mathrm{id}_M\otimes\mathrm{id}_A^{\otimes i}\otimes\eta\otimes\mathrm{id}_A^{\otimes n-i}\otimes\mathrm{id}_N$.

Proposition 5.9 applied to this context gives a natural simplicial homotopy equivalence

$$B_\bullet(A, A, N) \to N_\bullet$$

where $N_\bullet$ is the constant simplicial R-module at N. Compare with the result obtained in proposition (2.1.2).

5.4 The case of modules over an S-algebra

The case of modules over a commutative S-algebra R is formally analogous to the case of modules over a commutative ring k. In section 5.3, just replace k by R and $\otimes_k$ by $\wedge_R$. So here A is an R-algebra, M is a right A-module and N is a left A-module.

The bar construction $B^R_\bullet(M, A, N)$ yields a simplicial R-module. We can thus apply geometric realization to it, and we write

$$B^R(M, A, N) := |B^R_\bullet(M, A, N)|.$$

Proposition 5.10. *There is a natural homotopy equivalence of R-modules*

$$B^R(A, A, N) \to N.$$

Proof. Proposition 5.9 gives a simplicial homotopy equivalence $B^R_\bullet(A, A, N) \to N_\bullet$, and since geometric realization preserves homotopies (proposition 4.2) we get the result in view of remark 4.1. □

Remark 5.11. If M is a cell A-module, then there is a weak equivalence of R-modules $B^R(M, A, N) \to M \wedge_A N$. ([1, IX.2.3]) Compare with proposition 2.1.3.

Proposition 5.12. *Let $f : M \to M'$ be a morphism between right R-modules. Suppose it is a homotopy equivalence (resp. weak equivalence). Then the induced map on geometric realizations $B^R(M, A, N) \to B^R(M', A, N)$ is a homotopy equivalence (resp. weak equivalence).*

Proof. It is an application of proposition 4.4, since the hypothesis implies that $M \wedge A^n \wedge N \to M' \wedge A^n \wedge N$ is a homotopy equivalence (resp. weak equivalence) for all n. □

6 Topological Hochschild homology

We will now translate the definition(s) of Hochschild homology to the topological context.

Let R be a commutative ring spectrum, A be an R-algebra and M be an (A, A)-bimodule.

We denote $\wedge = \wedge_R$, the smash product in the category of R-modules.

6.1 As a simplicial spectrum

In section 3.1.1 we expressed Hochschild homology as (homotopy groups of) a simplicial k-module $\mathcal{H}_\bullet(A, M)$. We will now mimic that definition.

Let $\phi : A \wedge A \to A$ be the product of A and $\eta : R \to A$ be its unit. Let $\xi_\ell : A \wedge M \to M$ and $\xi_r : M \wedge A \to M$ be the left and right actions of A on M respectively.

Denote by $\tau : (M \wedge A^{\wedge n-1}) \wedge A \to A \wedge (M \wedge A^{\wedge n-1})$ the canonical isomorphism. We define the simplicial R-module $thh^R_\bullet(A, M)$ as $thh^R_n(A, M) = M \wedge A^{\wedge n}$ with face maps $d_i : M \wedge A^{\wedge n} \to M \wedge A^{\wedge n-1}$ and degeneracies $s_i : M \wedge A^{\wedge n} \to M \wedge A^{\wedge n+1}$ given by

$$d_i = \begin{cases} \xi_r \wedge \mathrm{id}_A^{\wedge n-1} & \text{if } i = 0, \\ \mathrm{id}_M \wedge \mathrm{id}_A^{\wedge i-1} \wedge \mu \wedge \mathrm{id}_A^{\wedge n-i-1} & \text{if } 0 < i < n, \\ (\xi_l \wedge \mathrm{id}_A^{\wedge n-1}) \circ \tau & \text{if } i = n \end{cases}$$

and $s_i = \mathrm{id}_M \wedge \mathrm{id}_A^{\wedge i} \wedge \eta \wedge \mathrm{id}_A^{\wedge n-i}$. We now define an R-module:

$$thh^R(A, M) := |thh^R_\bullet(A, M)|.$$

Proposition 6.1. *There is a natural map of R-modules $M \to thh^R(A, M)$.*

Proof. Consider the maps

$$M \cong M \wedge R^n \xrightarrow{\mathrm{id}\wedge\eta^n} M \wedge A^n .$$

These maps define a natural map of simplicial R-modules $M_\bullet \to thh^R_\bullet(A, M)$ where $M_\bullet$ is the constant simplicial R-module at M.

By applying geometric realization and remark 4.1, the statement follows. □

6.2 As a derived smash product

In section 3.2 we expressed Hochschild homology as $\mathrm{Tor}_*^{A^e}(M, A)$ when A was flat as a k-module. On the topological context, the analogous condition we need to impose on the commutative R-module A for the analogous expression to hold is that A be cofibrant in the model category of commutative R-modules. We will also need to suppose that R is cofibrant as an S-module. We make these assumptions in this section.

Define A^e to be $A \wedge A^{op}$. We define an R-module

$$THH^R(A, M) := M \overset{L}{\wedge}_{A^e} A$$

where the L on top of $\wedge$ denotes that we are working in the derived category of R-modules.

We also define

$$THH^R_*(A, M) := \pi_*(THH^R(A, M)).$$

For the above definitions to work how we want them to, we have to take M to be a cell A^e-module. This is the analogous condition of having to take a projective resolution of the A-module M to compute Tor in the algebraic case.

6.3 Relationship of thh with THH

The following proposition is proven in a formally analogous way to the corresponding statement for Hochschild homology (5), i.e. by first establishing a natural isomorphism $thh^R(A, M) \cong M \wedge_{A^e} B^R(A, A, A)$ where the last term is the bar construction of section 5.4.

Proposition 6.2. *If M is a cell A^e-module, there is a natural weak equivalence of R-modules*

$$thh^R(A, M) \to THH^R(A, M).$$

6.4 Relationship with Hochschild homology

Proposition 6.3. *Let k be a commutative ring, A be a k-flat k-algebra, and M be an (A, A)-bimodule. We have*

$$\mathcal{H}_*(A, M) \cong THH^{Hk}_*(HA, HM)$$

where H denotes Eilenberg-Mac Lane spectra.

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