MONOIDS

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Contents

1. Introduction

The aim of this note is to motivate the definition of a symmetric monoidal ∞-category. We will start by putting the definition of a classical symmetric monoidal category in a guise more suitable to generalization to the ∞-setting.

Recall the microcosm principle [BD98]:

> *Certain algebraic structures can be defined in any category equipped with a categorified version of the same structure.*

The prototypical example is given by monoids. We concentrate on the symmetric case. We call a monoid *symmetric* if it is commutative, in order to highlight the connection with symmetric monoidal categories. So, in this example, the algebraic structure of a "symmetric monoid" can be defined in any category which is already a symmetric monoid itself, i.e. a symmetric monoidal category. Note that this reasoning can be applied to itself: a (small) symmetric monoidal category itself is nothing but a symmetric pseudomonoid (= weak 2-monoid) in the (cartesian) symmetric monoidal 2-category of small categories **Cat**. For a reference for 2-categorical stuff, see [Lac10], [AM10, Appendix C] and [McC00].

The bottom line is: in order to understand symmetric monoidal categories, we might as well start by understanding symmetric monoids. We start by giving an "unbiased" version of a symmetric monoid. It should be intuitively easy to grasp what it is, but we also want a definition which is both concrete and theoretically illuminating. The most familiar path is perhaps through the theory of monads, but it will be more fruitful to use the theory of operads. We will see that the operad of commutative monoids is naturally linked to the category of finite sets, and then to Segal's category Γ. The conclusive remark is that a symmetric monoid in a *cartesian* monoidal category $\mathcal{C}$ is equivalently a functor $\Gamma^{\mathrm{op}} \to \mathcal{C}$ that satisfies that some "Segal maps" are isomorphisms.

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This is easily relaxed to a notion of a "symmetric ∞-monoid", provided $\mathcal{C}$ is a cartesian ∞-category: we now consider an ∞-functor $N(\Gamma^{\mathrm{op}}) \to \mathcal{C}$ such that the Segal maps are equivalences in $\mathcal{C}$. Taking $\mathcal{C} = \mathbf{Top}$ essentially gives Segal's approach to defining "special Γ-spaces", which are a model for E_∞-spaces (i.e. symmetric ∞-monoids).

This can be adapted to the categorical setting. **Cat** is only a 2-category, not an ∞-category like **Top**, so what we get is that a symmetric 2-monoid, i.e. a symmetric monoidal category, is equivalently a (weak) 2-functor $\Gamma^{\mathrm{op}} \to \mathbf{Cat}$ such that the Segal maps are equivalences of categories (the 2-cells in Γ^{op} are identities).

Finally, this also works in the realm of ∞-categories: letting $\mathbf{Cat}_\infty$ be the ∞-category of ∞-categories, a symmetric monoidal ∞-category will be an ∞-functor $N(\Gamma^{\mathrm{op}}) \to \mathbf{Cat}_\infty$ satisfying that the Segal maps are equivalences of ∞-categories.

We'd rather not mess with $\mathbf{Cat}_\infty$, so instead of taking this last step, what we realize is that a weak 2-functor $\Gamma^{\mathrm{op}} \to \mathbf{Cat}$ is equivalently a Grothendieck opfibration over Γ^{op}, and the Segal condition translates easily. This is the reformulation of a symmetric monoidal category which most easily adapts to the ∞-setting. Lurie calls the analog of a Grothendieck opfibration in the ∞-world a "coCartesian fibration": a symmetric monoidal ∞-category will be defined as a coCartesian fibration $\mathcal{C} \to \Gamma^{\mathrm{op}}$ such that some Segal maps are categorical equivalences (i.e. equivalences of ∞-categories).

2. Unbiased symmetric monoids

Let M be a symmetric monoid, i.e. a set M with a unit $\mu_0 : * \to M$ and a product $\mu_2 : M \times M \to M$, such that the associativity, unitality and commutativity axioms are satisfied.

The first remark we should make is that this is a *biased* definition. Let us explain this.

The structure of a monoid is sufficient to define a unique operation $\mu_n : M^{\times n} \to M$ such that μ_n is obtained by any kind of parenthesizing, adding units, or switching the elements given. This is easily proven by induction, and it is an easy instance of a *coherence theorem*. So μ_0 is to be seen as a 0-ary operation, giving the unit, $\mu_1 : M \to M$ is the identity, μ_2 is the binary product, and generally μ_n is the unique n-ary product.

In fact, it is equivalent to define a monoid by the biased definition given above, or by an *unbiased* definition which doesn't single out the arities 0 and 2 and the associativity, unitality and commutativity axioms, but instead gives all the operations and all the axioms.

It is instructive to define what an unbiased symmetric monoid really is. Consider the monad arising from the free-forgetful adjunction

$$\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{CMon}. \tag{2.1}$$

Explicitely, the monad is

$$T = U \circ F : \mathbf{Set} \to \mathbf{Set}, \quad X \mapsto \coprod_{n \in \mathbb{N}} X^{\times n} / \Sigma_n,$$

where Σ_n is the n-th symmetric group.

An algebra for T is a set X together with a map $TX \to X$ which is associative and unital. Note that such a map amounts to maps $X^{\times n}/\Sigma_n \to X$, i.e. Σ_n-equivariant maps

$$\mu_n : X^{\times n} \to X.$$

If one unwinds the definition of all of this, one gets that this is exactly what an unbiased symmetric monoid is expected to be, and the statement that it is the same as a standard monoid is the statement that there is a natural equivalence of categories T-$\mathbf{Alg} \to \mathbf{CMon}$ (i.e., that the adjunction (2.1) is monadic).[1]

The monad T comes from an operad. A *(symmetric) operad* (in **Set**) is a collection $\mathcal{P} = \{P(n)\}_{n\in\mathbb{N}}$ of sets, together with a unit object $1 \to P(1)$, and functions

$$P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(k_1 + \cdots + k_n)$$

for each $n, k_1, \ldots, k_n \geq 0$ satisfying unit and associativity axioms, plus a right action of Σ_n on $P(n)$, satisfying some compatibility conditions. See [MSS02, Definition II.1.4] or [May97] for a complete definition.

An *algebra* for $\mathcal{P}$ in a (cocomplete, closed) symmetric monoidal category $\mathcal{V}$ is an object $A \in \mathcal{V}$ and compatible functions

$$P(n) \odot_{\Sigma_n} A^{\otimes n} \to A$$

[MSS02, Definition II.1.20], where

$$P(n) \odot_{\Sigma_n} A^{\otimes n} \coloneqq \operatorname*{colim}_{\sigma\in\Sigma_n} \left(\coprod_{m\in P(n)} A^{\otimes n} \to \coprod_{m\in P(n)} A^{\otimes n} \right) \in \mathcal{V}$$

with the maps essentially being $(m, a_1 \otimes \cdots \otimes a_n) \mapsto (\sigma^{-1} \cdot m, a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)})$. One should think of the set $P(n)$ as parametrizing how many n-ary operations there are in a $\mathcal{P}$-algebra.

There is a category of algebras over $\mathcal{P}$ in $\mathcal{V}$, denoted $\mathcal{P}$-$\mathbf{Alg}(\mathcal{V})$. If $\mathcal{V} = \mathbf{Set}$ we suppress it from the notation.

Let $\mathcal{C}om$ be the operad with $P(n) = 1$ for all $n \geq 0$. A $\mathcal{C}om$-algebra is equivalently an algebra over the monad T (adapted to $\mathcal{V}$), i.e. an unbiased symmetric monoid, because $1 \odot_{\Sigma_n} A^{\otimes n} = A^{\otimes n}/\Sigma_n$. More generally, any operad in $\mathcal{V}$ gives rise to a monad in $\mathcal{V}$, but we won't be needing this.[2]

We now fix the definition of an unbiased symmetric monoid once and for all:

Definition 2.2. An *unbiased symmetric monoid* in $\mathcal{V}$ is a $\mathcal{C}om$-algebra in $\mathcal{V}$.

This can be spelled out without too much hassle. For example, the following diagram encodes both unitality and associativity (the latter is obtained in an indirect way, namely, by encoding that both traditional ways of going from three factors to one factor by parenthesizing coincide with the given three-fold way of going from three factors to one directly). So denote by $\mu_n : A^{\times n} \to A$ the Σ_n-equivariant structure maps of the unbiased monoid A. If $j = j_1 + \cdots + j_k$, the following diagram has to commute.

$$\begin{array}{ccc} A^j & \xrightarrow{\mu_j} & A \\ {\scriptstyle\cong}\downarrow & & \uparrow{\scriptstyle\mu_k} \\ A^{j_1} \times \cdots \times A^{j_k} & \xrightarrow[\mu_{j_1}\times\cdots\times\mu_{j_k}]{} & A^k \end{array}$$

We have given the definition of an operad in **Set** for simplicity, but note that one can define an operad in any symmetric monoidal category $\mathcal{C}$, and there are algebras for that operad in any $\mathcal{C}$-enriched symmetric monoidal category $\mathcal{V}$.

[1]Note that this also gives that an *unbiased morphism* of unbiased monoids coincides with morphisms of monoids.

[2]It is $\mathcal{V} \to \mathcal{V}$, $X \mapsto \coprod_{n\in\mathbb{N}} P(n) \otimes_{\Sigma_n} X^{\otimes n}$.

3. The prop associated to an operad

[Lei00, Section 1.6] Let $\mathcal{V}$ be a cocomplete closed symmetric monoidal category.

Theorem 3.1. *Let $\mathcal{P}$ be an operad. There exists a symmetric strict[3] monoidal category $\widehat{\mathcal{P}}$ and an equivalence of categories*

$$\mathcal{P}\text{-}\mathbf{Alg}(\mathcal{V}) \simeq \mathbf{StrongSym}(\widehat{\mathcal{P}}, \mathcal{V})$$

where **StrongSym** *denotes strong symmetric monoidal functors.*

There are several ways to construct $\widehat{\mathcal{P}}$. Let us concentrate on the $\mathcal{P}$ that matters to us, i.e. $\mathcal{C}om$. Then

$$\widehat{\mathcal{C}om} = \mathbf{Fin},$$

and therefore

$$\mathcal{C}om\text{-}\mathbf{Alg}(\mathcal{V}) \simeq \mathbf{StrongSym}(\mathbf{Fin}, \mathcal{V}). \tag{3.2}$$

Fin is a skeleton of the category of finite sets. Its objects are the natural numbers $\mathbf{n} = \{0, \dots, n-1\}$, $n \geq 0$ (for $n = 0$ this is empty), and the morphisms are the functions between those sets. The monoidal product of $\mathbf{m}$ and $\mathbf{n}$ is $\mathbf{m} + \mathbf{n} = \{0, \dots, m+n-2\}$, and its unit is 0. The symmetry is the map $\mathbf{m} + \mathbf{n} \to \mathbf{n} + \mathbf{m}$ which you would expect. Thus, it is a symmetric strict monoidal category. See [Gra01, Section 2]

A general procedure to build $\widehat{\mathcal{P}}$ is as the "free symmetric strict monoidal category containing a $\mathcal{P}$-algebra". So the claim is that **Fin** is the free symmetric strict monoidal category containing a symmetric monoid. The symmetric monoid in question is the finite set $\mathbf{1}$ with unit $\mathbf{0} \to \mathbf{1}$ and multiplication $\mathbf{2} \to \mathbf{1}$. It might be more instructive to see how to associate to an unbiased symmetric monoid in $\mathcal{V}$, i.e. a $\mathcal{C}om$-algebra in $\mathcal{V}$, a strong symmetric monoidal functor $\mathbf{Fin} \to \mathcal{V}$, so let us do this.

For simplicity, suppose $\mathcal{V} = \mathbf{Set}$ and let A be an unbiased symmetric monoid. We have maps $\mu_n : A^{\times n}/\Sigma_n \to A$, which take any set of n elements in A and give you an element of A (the order of the elements does not matter, so we might as well consider that μ_n takes as input a set of n elements).

Define a strong symmetric monoidal functor $\mathbf{Fin} \to \mathbf{Set}$ given by $\mathbf{n} \mapsto A^{\times n}$, and if $f : \mathbf{n} \to \mathbf{m}$ is a function, then the function $A^{\times n} \to A^{\times m}$ is given by

$$(a_1, \dots, a_n) \mapsto \left(\mu_{\# f^{-1}(0)}\{a_i : i \in f^{-1}(0)\}, \dots, \mu_{\# f^{-1}(m-1)}\{a_i : i \in f^{-1}(m-1)\}\right).$$

It is perhaps clearer to express this as

$$(a_1, \dots, a_n) \mapsto \left(\prod_{i \in f^{-1}(0)} a_i, \dots, \prod_{i \in f^{-1}(m-1)} a_i \right) \tag{3.3}$$

where an empty product denotes a unit.

Conversely, if $F : \mathbf{Fin} \to \mathbf{Set}$ is a strong symmetric monoidal functor, let $A = F(\mathbf{1})$. Then $F(\mathbf{1})$ is an unbiased symmetric monoid: let $\mathbf{n} \to \mathbf{1}$ be the unique map, so when we apply F we get $F(\mathbf{n}) \cong F(\mathbf{1})^n = A^n \to A$, the isomorphism gotten because F is strong and $\mathbf{n} = \mathbf{1} + \cdots + \mathbf{1}$.

Side remark 3.4. There is a non-commutative analog for all of this. One can consider non-symmetric operads (with no symmetric group actions), and algebras for these. Now $\mathcal{A}ss$ is the non-symmetric operad having one element in each degree. Algebras over $\mathcal{A}ss$ are unbiased monoids. The prop for $\mathcal{A}ss$ is now **Ord**, the category of finite ordinals and order-preserving functions (i.e., it is the free strict monoidal category with a monoid). Just as the morphisms

[3] The wording is carefully chosen to suggest that we are not imposing that $A \otimes B = B \otimes A$.

of **Ord** are generated by certain coface and codegeneracy maps, the morphisms of **Fin** are generated by coface, codegeneracy, and main transpositions, see [Gra01, Section 3].

4. The cartesian scenario

[Lei00, Section 3.1] Suppose that $\mathcal{V}$ is now cartesian, so we denote it by $\mathcal{C}$. This means that the unit of $\mathcal{V}$ is a terminal object, and that the monoidal product is a categorical product. It turns out that in this case, we have yet another formulation of an unbiased symmetric monoid.

Definition 4.1. Γ is the opposite category of $\mathbf{Fin}_*$, where $\mathbf{Fin}_*$ is a skeleton of the category of finite pointed sets.[4] Concretely, the objects of $\mathbf{Fin}_*$ are $[n] = \{0, \dots, n\}$, $n \geq 0$, and maps are functions mapping 0 to 0. Note that, with our notation, both $\mathbf{n} \in \mathbf{Fin}$ and $[n-1] \in \mathbf{Fin}_*$ are sets with n elements.

Remark 4.2. I think it was Segal [Seg74] who introduced Γ, though actually he didn't identify it as the opposite category of $\mathbf{Fin}_*$ (I think the identification is due to Anderson [And71, Section 3]). Instead, he described it directly: its objects are the $\mathbf{n}$, $n \in \mathbb{N}$, and a map $\mathbf{n} \to \mathbf{m}$ in Γ is a function $\mathbf{n} \to \mathcal{P}(\mathbf{m})$ such that the images of two different i are disjoint. The composition $\mathbf{n} \xrightarrow{f} \mathbf{m} \xrightarrow{g} \mathbf{p}$ maps i to $\bigcup_{j \in f(i)} g(j)$. The equivalence of this category with $\mathbf{Fin}_*^{\mathrm{op}}$ corresponds $\mathbf{n}$ with $[n]$.

Note as well that $\mathbf{Fin}_*$ is equivalent to a skeleton of the category of finite sets with partially defined functions (i.e. functions defined on a subset of the domain).

Theorem 4.3. *There is an isomorphism of categories*

$$\mathbf{SColax}(\mathbf{Fin}, \mathcal{C}) \cong \mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{C}),$$

where **SColax** *denotes colax symmetric monoidal functors.*

Under this isomorphism, the functors $\mathbf{Fin} \to \mathcal{C}$ *which are strong correspond to the functors* $Y : \Gamma^{\mathrm{op}} \to \mathcal{C}$ *satisfying the following equivalent conditions:*

(1) For all $m, n \geq 0$*, the map in* $\mathcal{C}$

$$(Y(\pi_1), Y(\pi_2)) : Y[m+n] \to Y[m] \times Y[n] \tag{4.4}$$

is an isomorphism. The maps $\pi_1 : [m+n] \to [m]$ *and* $\pi_2 : [m+n] \to [n]$ *are the obvious ones.*

(2) For all $n \geq 0$*, the following* Segal map *in* $\mathcal{C}$

$$(Y(\rho_1), \dots, Y(\rho_n)) : Y[n] \to Y[1]^n \tag{4.5}$$

is an isomorphism, where $\rho_j : [n] \to [1]$*,* $j = 1, \dots, n$ *maps* j *to* 1 *and everything else to* 0*.*

Proof. Let $X : \mathbf{Fin} \to \mathcal{C}$ be colax, with structure map ∇. We associate to it the functor $Y : \Gamma^{\mathrm{op}} \to \mathcal{C}$ which we now define. Set $Y[n] = X(\mathbf{n})$. Now let $\eta_m : \mathbf{1} + \mathbf{m}$ in **Fin** be the map $i \mapsto 1 + i$. If $g : [m] \to [n]$ is in Γ^{op}, define $\mathbf{g} : \mathbf{1} + \mathbf{m} \to \mathbf{1} + \mathbf{n}$ in **Fin** obviously. For such a g, define $Y(g)$ as the composition

$$X(\mathbf{m}) \xrightarrow{X(\eta_m)} X(\mathbf{1} + \mathbf{m}) \xrightarrow{X(\mathbf{g})} X(\mathbf{1} + \mathbf{n}) \xrightarrow{\nabla} X(\mathbf{1}) \times X(\mathbf{n}) \xrightarrow{\pi_2} X(\mathbf{n}).$$

Now let $Y : \Gamma^{\mathrm{op}} \to \mathcal{C}$ be any functor. We associate to it a functor $X : \mathbf{Fin} \to \mathcal{C}$, given by $X(\mathbf{n}) = Y[n]$. If $f : \mathbf{m} \to \mathbf{n}$ is in **Fin**, there is an obvious associated map $[f] : [m] \to [n]$ in

[4] Leinster's opinion is that one shouldn't think too hard about Γ^{op} being obtained by pointing the objects of **Fin**, which is the prop for $\mathcal{C}om$. He suggests that this line of thought is a "distracting coincidence", or a "red herring". See Side Remark 4.9.

Γ^{op}, so define $X(f) = Y[f]$. As for the colax structure, the counit $X(\mathbf{0}) \to \mathbb{1}$ is the unique such map. The structure map $X(\mathbf{m}+\mathbf{n}) \to X(\mathbf{m}) \times X(\mathbf{n})$ is given by (4.4).

It is obvious that a colax $X : \mathbf{Fin} \to \mathcal{C}$ is strong if and only if its associated Y satisfies that (4.4) is an isomorphism. It remains to prove that conditions (1) and (2) are equivalent. To prove that (1) implies (2), note that the Segal map can be obtained as the composition

$$Y[n] \to Y[n-1] \times Y[1] \to Y[n-2] \times Y[1] \times Y[1] \to \cdots \to Y[1]^n,$$

so as a composite of isomorphisms, it is an isomorphism. Conversely, note that the Segal map $Y[m+n] \overset{\cong}{\to} Y[1]^{m+n} \cong Y[1]^m \times Y[1]^n$ splits as a composition

$$Y[m+n] \to Y[m] \times Y[n] \overset{\cong}{\to} Y[1]^m \times Y[1]^n$$

where the latter map is a product of two Segal maps, so by the 2-out-of-3 property for isomorphisms, the map $Y[m+n] \to Y[m] \times Y[n]$ is an isomorphism. □

We will use the second condition more often. Note that when $n = 0$, this means that we have an isomorphism $Y[0] \to \mathbb{1}$.

Putting together this theorem with (3.2) we get:

Corollary 4.6. *There is an equivalence of categories between unbiased symmetric monoids in $\mathcal{C}$ and functors $\Gamma^{\mathrm{op}} \to \mathcal{C}$ satisfying that the Segal maps are isomorphisms.*

In the case $\mathcal{C} = \mathbf{Set}$, if A is a symmetric monoid then its associated functor $Y : \Gamma^{\mathrm{op}} \to \mathbf{Set}$ is $[n] \mapsto A^n$, and $f : [n] \to [m]$ goes to the function $A^n \to A^m$ given by

$$(a_1, \ldots, a_n) \mapsto \left(\prod_{i \in f^{-1}(1)} a_i, \ldots, \prod_{i \in f^{-1}(m)} a_i \right), \tag{4.7}$$

compare with (3.3). Define

$$m_n : [n] \to [1], \qquad 0 \mapsto 0, i \mapsto 1 \text{ when } i \neq 0, \tag{4.8}$$

then Y maps m_2 to the binary multiplication of A, and the higher m_n to the higher multiplication of A. The only map $[0] \to [1]$ maps to $1 \to A$, the unit map of A. The Segal maps are identities.

Side remark 4.9. Just as in Remark 3.4, there is a non-commutative analog. The place of Γ is now taken by Δ, and as in Footnote 4, Leinster's opinion is that one shouldn't think too hard about Δ^{op} being obtained as the opposite category of the category obtained by removing the empty set from **Ord**.

A generalization of Theorem 4.3 is in [Lei00, Proposition 3.1.5]. Γ^{op} is then revealed as the Kleisli category for the monad $\mathbf{1} \times -$ on **Fin**, and Δ^{op} as the Kleisli category for the monad $\mathbf{1} \times - \times \mathbf{1}$ on **Ord**.

Barwick [Bar17] studies the passage from **Fin** (resp. **Ord**) to Γ^{op} (resp. Δ^{op}) more systematically. He calls the latter the *Leinster category* of the former. He writes that the corresponding monads "add points to any object in all the ways one can do so functorially".

Actually, this passage had already been obtained (in a different form) by [MT78] (resp. [Tho79]): Γ^{op} (resp. Δ^{op}) is the *category of operators* for the operad $\mathcal{C}om$ (resp. the non-symmetric operad $\mathcal{A}ss$). This can be characterized as the free semi-cartesian monoidal category on the free semi-cartesian operad associated to that operad. See [Shu]. In [MT78, Definition 1.2], the notion of a $\hat{C}$-space where $\hat{C}$ is the category of operators of an operad C is studied, and it essentially coincides with the notion of a "homotopy algebra" of [Lei00].

5. Weakening

Now suppose $\mathcal{C}$ has a subcategory of weak equivalences. Then a definition for a symmetric ∞-monoid in $\mathcal{C}$ could be a functor $\Gamma^{\mathrm{op}} \to \mathcal{C}$ such that the Segal maps are weak equivalences. We promote this to a definition:

Definition 5.1. Let $\mathcal{C}$ be a cartesian category with weak equivalences. We define a Γ-*object* in $\mathcal{C}$ to be a functor $\Gamma^{\mathrm{op}} \to \mathcal{C}$, and a *special* Γ-*object* to be a Γ-object Y such that the Segal maps

$$(Y(\rho_1), \dots, Y(\rho_n)) : Y[n] \to Y[1]^n \tag{5.2}$$

are weak equivalences.

Remark 5.3. We have deliberately not defined the concept of a "cartesian category with weak equivalences" – the class of weak equivalences should definitely satisfy 2-out-of-3, and a cartesian monoidal model category should, in any case, be a source of examples.

In the proof of the equivalence of conditions (1) and (2) in Theorem 4.3, the only thing we used about the class of isomorphisms was that it satisfies 2-out-of-3. Therefore, a Γ-object is special if and only if the maps (4.4) are weak equivalences, though we shall not be using this.

This terminology is classical: I think it was introduced in [And71] following Segal, and was then adopted in [BF78] (note that [Seg74] calls "Γ-object" what we call a "special Γ-object").

Example 5.4. The archetypical example for $\mathcal{C}$ is **Top** (say, compactly generated weakly Hausdorff spaces) with the class of weak equivalences being the homotopy equivalences. Then special Γ-spaces were an early model of E_∞-spaces, i.e. symmetric ∞-monoids.

If Y is a special Γ-space, then there is a weak equivalence $Y[2] \to Y[1] \times Y[1]$, so speaking informally, "the multiplication of two elements in Y is given by

$$Y[1] \times Y[1] \xleftarrow[(Y(\rho_1),Y(\rho_2))]{\sim} Y[2] \xrightarrow[Y(m_2)]{} Y[1] \text{ ''}. \tag{5.5}$$

This is not an honest map, "just as" there is no chosen composition of arrows in an ∞-category. To get an honest map, one should pick a homotopy inverse for $(Y(\rho_1), Y(\rho_2))$.

Side remark 5.6. Taking π_0 of (5.5) shows that $\pi_0 Y[1]$ is a symmetric monoid under the operation

$$\pi_0 Y[1] \times \pi_0 Y[1] \xleftarrow{\cong} \pi_0 Y[2] \longrightarrow \pi_0 Y[1] \ .$$

If $\pi_0(Y)$ has an inverse, i.e. if it is an abelian group, we say that Y is a *very special* Γ-*space*. These are a model for grouplike E_∞-spaces, infinite loop spaces and connective spectra. But this doesn't go in the direction we want to head.

Side remark 5.7. Leinster [Lei00, Theorem 3.1.2] proves that, under the correspondence of Theorem 4.3, the colax symmetric monoidal functors whose structure maps are weak equivalences coincide with the special Γ-objects, *provided* we work in a setting [Lei00, Definition 2.1.1] which, in fact, is quite restrictive. It turns out that for a monoidal model category to fit in this setting, all objects have to be cofibrant (since in a general monoidal model category, the tensor product of two weak equivalences is a weak equivalence *provided* enough of the objects concerned are cofibrant). Leinster used this theorem to motivate the definition of a symmetric ∞-monoid (which he calls a "homotopy $\mathcal{C}om$-algebra") in any symmetric monoidal (not necessarily cartesian) category $\mathcal{V}$ as a colax symmetric monoidal functor $\mathbf{Fin} \to \mathcal{V}$ whose structure maps are weak equivalences, but as we have seen, the setting is too restrictive (more accepted definitions of ∞-algebras for ∞-operads can be found in [Lur12]). At any rate, once more, this is not the path we want to take: we'll stick to the definition involving Γ-objects.

6. Symmetric monoidal categories, revisited

We have recast the definition of an unbiased symmetric monoid in a cartesian monoidal category $\mathcal{C}$ as a Γ-object in $\mathcal{C}$ such that its Segal maps (4.5) are isomorphisms. By the general nonsense explained in the introduction, this should help in formulating a definition of an unbiased symmetric monoidal category.

The problem is that a symmetric monoidal category is not really a monoid, but a symmetric *pseudomonoid* (or *weak monoid*) in the 2-category Cat. Indeed, the definition of a symmetric monoidal category uses the 2-cells in Cat: we do not ask for example that $A \otimes (B \otimes C)$ be *equal* to $A \otimes (B \otimes C)$: instead, we'll have the component of a natural isomorphism between them.

What should an unbiased symmetric monoidal category be? It should be a weak algebra for $\mathcal{C}om$ in **Cat** (let us sweep size issues under the rug). In fact, there is a theory of weak algebras for 2-monads (and I guess an analogous thing must exist for 2-operads in monoidal bicategories), it does give the right thing, and one can write down explicitly what it gives. See [Lei04, Section 3.1] for details. One specifies a category $\mathcal{V}$, operations $\mathcal{V}^{\times n}/\Sigma_n \to \mathcal{V}$ and a ton of associators and unitors satisfying a lot of things.

Is this the same as a symmetric monoidal category? In the monoid case, this theorem was simple: the fact that we can prove e.g. $(ab)(cd) = a((bc)d)$ follows easily from standard associativity. Here, it's subtler, and it's exactly the content of Mac Lane's coherence theorem, which states that the small list of axioms given for a symmetric monoidal category satisfy to prove that everything which should agree, agrees. [5] [Lei04, Section 3.1] gives a great exposition of this.

So here's how we proceed:

0) The 2-category of symmetric monoidal categories and strong symmetric monoidal functors is equivalent to the 2-category of unbiased symmetric monoidal categories, which are weak $\mathcal{C}om$-algebras in **Cat**,
1) which is equivalent to some 2-category of weak monoidal pseudofunctors $\mathbf{Fin} \to \mathbf{Cat}$,
2) which is equivalent to the 2-category of pseudofunctors $\Gamma^{\mathrm{op}} \to \mathbf{Cat}$ such that the Segal maps are equivalences of categories.

I have explained (0). I won't explain (1), but it is an intermediate step we do not need, and I haven't thought about it deeply; at any rate, the correct notion of a weak monoidal pseudofunctor should be somewhere in [McC00, Section 4]. As for (2), it is easy to believe. Note that this is not exactly about special Γ-categories, for such things are merely functors (not pseudofunctors). [6] Here we are considering Γ^{op} as a 2-category with only identity 2-cells.

Summing up, what matters is:

Theorem 6.1. *There is an equivalence of 2-categories between symmetric monoidal categories with strong symmetric monoidal functors and the 2-category of pseudofunctors* $\Gamma^{\mathrm{op}} \to \mathbf{Cat}$ *such that the Segal maps (5.2) are equivalences of categories.*

So if $\mathcal{V}$ is a symmetric monoidal category, then its associated $Y : \Gamma^{\mathrm{op}} \to \mathbf{Cat}$ takes $[n]$ to $\mathcal{V}^{\times n}$, takes $m_2 : [2] \to [1]$ (4.8) to the tensor product functor, $[0] \to [1]$ to the unit, and the

[5] Note that there is an analog theorem for symmetric monoidal functors, proving that an unbiased monoidal functors (of whichever type) correspond to monoidal functors (of the same type).

[6] The notion of (3) could be called *2-special* Γ*-category*, since it uses the 2-categorical structure of **Cat**. Note that special Γ-categories *are* interesting, only for other purposes. E.g. the levelwise nerve of a special Γ-category is a special Γ-space (or rather, Γ-simplicial set), and this is extremely fruitful, because every symmetric strict monoidal category (a "permutative category") gives rise to a special Γ-category via a "K-theory" functor, see [Seg74] and [Man10].

higher $m_n : [n] \to [1]$ to the higher tensor product functors. More generally, if $f : [n] \to [m]$ in Γ^{op}, then $Y(f) : V^{\times n} \to \mathcal{V}^{\times m}$ is a functor such that, on objects,

$$(A_1, \dots, A_n) \mapsto \left(\bigotimes_{i \in f^{-1}(1)} A_i, \dots, \bigotimes_{i \in f^{-1}(m)} A_i \right)$$

and defined obviously on arrows. Compare with (4.7) for the case of symmetric monoids in **Set**.

Note that this makes Y a *pseudofunctor* and not a functor. Indeed: take e.g.

$$[3] \xrightarrow{f} [2] \xrightarrow{m_2} [1]$$

where f is $i \mapsto i$ for $i = 0, 1, 2$ and $3 \mapsto 2$. The composition is m_3, hence $Y(m_3)$ is the triple product map $\mathcal{V}^{\times 3} \to \mathcal{V}$, and $Y(m_2) \circ Y(f)$ is the functor $\mathcal{V}^{\times 3} \to \mathcal{V}$ given on objects by $(A, B, C) \mapsto A \otimes (B \otimes C)$. These are not equal in general, only connected by a natural isomorphism. Note that by considering the pointed map $[3] \to [2]$ given by $1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2$ we recover the standard associator.

7. The Grothendieck construction

[Gro10, Section 4.1], [Vis05, Chapter 3].

We now describe a construction which allows us to rephrase the concept of a "pseudofunctor out of a 1-category" entirely in 1-categorical terms.

We first need some preliminary definitions.

Definition 7.1. Let $p : \mathcal{C} \to \mathcal{B}$ be a functor and $b \in \mathcal{B}$. The *fiber of p over b* is the category $\mathcal{C}_b$ fitting into the pullback

$$\begin{array}{ccc} \mathcal{C}_b & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \xrightarrow[b]{} & \mathcal{B}. \end{array}$$

More precisely, $\mathcal{C}_b$ has objects the $c \in \mathcal{C}$ such that $p(c) = b$, and the arrows are maps $c \to c'$ that map to id_b by p.

Definition 7.2. Let $p : \mathcal{C} \to \mathcal{B}$ be a functor and let $f : c_1 \to c_2$ be an arrow in $\mathcal{C}$. The arrow f is *p-coCartesian* if the following is a pullback diagram for all $c_3 \in \mathcal{C}$.

$$\begin{array}{ccc} \mathcal{C}(c_2, c_3) & \xrightarrow{f^*} & \mathcal{C}(c_1, c_3) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{B}(p(c_2), p(c_3)) & \xrightarrow[p(f)^*]{} & \mathcal{B}(p(c_1), p(c_3)). \end{array}$$

Let $\alpha = p(f) : b_1 \to b_2$. We say that f is a *p-coCartesian lift of α*.

Note that the pullback condition above can be rephrased as: the following functor is an isomorphism

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{B}_{p(c_1)/}} \mathcal{B}_{p(f)/}. \tag{7.3}$$

This can be rephrased more elementarily in terms of a lifting property for morphisms [Gro10, Proposition 4.2]. What we need is actually the following

Lemma 7.4. *Let $f : c_1 \to c_2$ and $f' : c_1 \to c_2'$ be two p-coCartesian lifts of $\alpha : b_1 \to b_2$. Then there is a unique isomorphism $\varphi : c_2 \to c_2'$ in $\mathcal{C}_{b_2}$ such that $\varphi \circ f' = f$.*

So p-coCartesian lifts with a fixed domain are essentially unique, when they exist. We consider existence in the following

Definition 7.5. A *Grothendieck opfibration over* $\mathcal{B}$ is a functor $p : \mathcal{C} \to \mathcal{B}$ such that for every $c_1 \in \mathcal{C}$ and every $\alpha : p(c_1) \to d$ in $\mathcal{D}$ there is a p-coCartesian lift $c_1 \to c_2$ of α. These form a category denoted $\mathrm{Opfib}(\mathcal{B})$.

For a Grothendieck opfibration $p : \mathcal{C} \to \mathcal{B}$, choosing a p-cocartesian lift for an arrow $\alpha : b_1 \to b_2$ in $\mathcal{B}$ (it is essentially unique by the above lemma) defines a functor

$$\alpha_! = \mathcal{C}_\alpha : \mathcal{C}_{b_1} \to \mathcal{C}_{b_2},$$

mapping c_1 to the codomain of any p-coCartesian lift of α. Using the lemma above one can prove that there is an essentially unique pseudofunctor

$$\mathcal{C}_{-} : \mathcal{B} \to \mathbf{Cat}, \quad b \mapsto \mathcal{C}_b, \quad (\alpha : b_1 \to b_2) \mapsto (\mathcal{C}_\alpha : \mathcal{C}_{b_1} \to \mathcal{C}_{b_2}). \tag{7.6}$$

One can go the other way, and associate to a pseudofunctor $\mathcal{B} \to \mathbf{Cat}$ a Grothendieck opfibration over $\mathcal{B}$:

Definition 7.7. Let $F : \mathcal{B} \to \mathbf{Cat}$ be a pseudofunctor. We define the *category of elements of* F, $\int F$. The objects of $\int F$ are pairs (b, a) where $b \in \mathcal{B}$ and $a \in Fb$. An arrow $(b, a) \to (b', a')$ consists of an arrow $f : b \to b'$ in $\mathcal{B}$ and an arrow $F(f)(a) \to a'$ in Fb'.[7]

Note that there is an obvious projection functor $\int F \to \mathcal{B}$ forgetting the second variable.

Proposition 7.8. [Vis05, 3.1.3] *The construction* $\int$ *is functorial. The projection* $\int F \to \mathcal{B}$ *is a Grothendieck opfibration over* $\mathcal{B}$*, and there is an equivalence of 2-categories*

$$\int : \mathrm{Pseudo}(\mathcal{B}, \mathbf{Cat}) \to \mathrm{Opfib}(\mathcal{B})$$

where Pseudo *denotes pseudofunctors. The inverse is given by choosing coCartesian lifts*[8] *and considering the fiber functor (7.6).*

It is instructive is to check what happens in the case of a symmetric monoidal category $\mathcal{V}$. Let $Y : \Gamma^{\mathrm{op}} \to \mathbf{Cat}$ be the pseudofunctor associated to it via Theorem 6.1. Then $\int Y$ is the category with objects $([n], (A_1, \ldots, A_n))$, where $[n] \in \Gamma^{\mathrm{op}}$ and $A_i \in \mathcal{V}$ for all i. We might as well just denote them by $(A_1, \ldots, A_n)$. A morphism is given by a map $f : [n] \to [m]$ in Γ^{op} and a map

$$\left(\bigotimes_{i \in f^{-1}(1)} A_i, \ldots, \bigotimes_{i \in f^{-1}(m)} A_i \right) \to (B_1, \ldots, B_m) \quad \text{in } \mathcal{V}.$$

We denote the category $\int Y$ by $\mathcal{V}^{\otimes}$.[9] It should be instructive to check explicitely what it means in this case for $\mathcal{V}^{\otimes} \to \Gamma^{\mathrm{op}}$ to be a Grothendieck opfibration.

[7] This is the 2-pullback of F along $\mathbf{Cat}_* \to \mathbf{Cat}$, where $\mathbf{Cat}_*$ is the 2-category of lax-pointed categories, i.e. its objects are pairs $(\mathcal{C}, c)$ where $\mathcal{C}$ is a category and $c \in \mathcal{C}$, and the morphisms are functors $F : \mathcal{C} \to \mathcal{C}'$ together with an arrow $F(c) \to c'$. The functor $\mathbf{Cat}_* \to \mathbf{Cat}$ is the obvious projection.

[8] I.e., choosing a *cleavage*, see [Vis05, Definition 3.9].

[9] Lurie [Lur12, Construction 2.0.0.1] constructs the same $\mathcal{V}^{\otimes}$, but his description is slightly different, because whereas we interpret a functor $\Gamma^{\mathrm{op}} \to \mathbf{Cat}$ as a covariant functor $\mathbf{Fin}_* \to \mathbf{Cat}$, he interprets is as a contravariant functor $\Gamma \to \mathbf{Cat}$ using the explicit description of Γ given in Remark 4.2.

Lemma 7.9. *Let $Y : \Gamma^{\mathrm{op}} \to \mathbf{Cat}$ be a pseudofunctor. It satisfies that the Segal maps (4.5) are equivalences of categories if and only if its corresponding Grothendieck opfibration $\mathcal{C} \to \Gamma^{\mathrm{op}}$ satisfies that the functor*

$$(\mathcal{C}_{\rho_1}, \dots, \mathcal{C}_{\rho_n}) : \mathcal{C}_{[n]} \to \mathcal{C}_{[1]}^{\times n} \tag{7.10}$$

is an equivalence of categories for all $n \geq 0$, where the $\rho_i : [n] \to [1]$, $i = 1, \dots, n$ were defined in (4.5).

Pasting the previous proposition with the previous lemma and with Theorem 6.1, we obtain the following

Corollary 7.11. *There is an equivalence of 2-categories between symmetric monoidal categories with strong symmetric monoidal functors and Grothendieck opfibrations with base Γ^{op} such that the functors (7.10) are equivalences of categories.*

For example, let $p : \mathcal{C} \to \Gamma^{\mathrm{op}}$ is a Grothendieck opfibration satisfying that the Segal maps (7.10) are equivalences. The symmetric monoidal category associated to it will be $\mathcal{C}_{[1]}$. The Segal condition for $n = 0$ says that $\mathcal{C}_{[0]}$ is equivalent to a category with a single object, and the only map $[0] \to [1]$ determines the unit object of $\mathcal{C}_{[1]}$. Using the map m_2 from (4.8), we get

$$\mathcal{C}_{[1]} \times \mathcal{C}_{[1]} \xleftarrow[(\mathcal{C}_{\rho_1}, \mathcal{C}_{\rho_2})]{\sim} \mathcal{C}_{[2]} \xrightarrow[m_2]{} \mathcal{C}_{[1]}$$

so choosing a pseudo-inverse for the left map we get a product map $\mathcal{C}_{[1]} \times \mathcal{C}_{[1]} \to \mathcal{C}_{[1]}$, well-defined up to canonical isomorphism, etc. (see [Lur12, Under Remark 2.0.0.6] for more worked-out details on how this gives a symmetric monoidal structure on $\mathcal{C}_{[1]}$). Compare with the Example 5.4 of special Γ-spaces. [10]

8. Symmetric monoidal ∞-categories

[Gro10, Section 4.2], [Lur09] and [Lur12, Section 2.0].

The definition of a symmetric monoidal ∞-category is an adaptation of the characterization given in Corollary 7.11 to the ∞-world. The analog of a Grothendieck opfibration will be a "coCartesian fibration" of ∞-categories. We adapt the definitions.

Definition 8.1. Let $p : X \to S$ be a map of simplicial sets and $s \in S$. The *fiber* of s under p is the simplicial set X_s given by the pullback

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \downarrow & & \downarrow p \\ \mathrm{pt} & \xrightarrow[s]{} & S. \end{array}$$

Definition 8.2. A map $p : X \to S$ of simplicial sets is an *inner fibration* if it has the right lifting property with respect to the inner horns $\Lambda^n_k \to \Delta^n$, $0 < k < n$:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & S. \end{array}$$

Remark 8.3. (1) By definition, the map $X \to \mathrm{pt}$ is an inner fibration if and only if X an ∞-category.

[10] Using the terminology from Footnote 6, special Γ-spaces model symmetric ∞-monoids in **Top**, and 2-special Γ-categories model symmetric (weak) 2-monoids in **Cat**.

(2) Any class of morphisms defined by a right lifting property is closed under base change, so inner fibrations are, too. I.e. if $S' \to S$ is a map of simplicial sets, then its pullback against p is an inner fibration $X' \to S'$. Combining this remark with the previous one, we get that the fiber of an arbitrary inner fibration is an ∞-category. Lurie [Lur09, Section 2.3] says: "we may therefore think of p as encoding a family of ∞-categories parametrized by S. However, the fibers X_s depend functorially on s in a very weak sense". This weak functoriality is what a co-Cartesian fibration, defined below, remedies.

(3) Given a functor $F : \mathcal{C} \to \mathcal{D}$ between ordinary categories, its nerve $NF : N\mathcal{C} \to N\mathcal{D}$ is automatically an inner fibration. Thus, the notion of inner fibration does not have a classical categorical counterpart.

Definition 8.4. Let $p : \mathcal{C} \to \mathcal{B}$ be a functor between ∞-categories. An arrow $f : c_1 \to c_2$ is *p-coCartesian*, or a *p-coCartesian lift of* $\alpha := p(f)$, if the following map is an acyclic Kan fibration:

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{B}_{p(c_1)/}} \mathcal{B}_{p(f)/}.$$

Definition 8.5. A functor $p : \mathcal{C} \to \mathcal{B}$ between ∞-categories is a *coCartesian fibration* if p is an inner fibration, and for all $c_1 \in \mathcal{C}$ and all $\alpha : p(c_1) \to b_2$ in $\mathcal{B}$ there exists a p-coCartesian lift $f : c_1 \to c_2$ of α.

Remark 8.6. [Gro10, Perspective 4.16] Just as in the classical case, Lurie proves that giving a coCartesian fibration $p : \mathcal{C} \to \mathcal{B}$ is equivalent to giving a functor (i.e. an ∞-functor!) $\mathcal{B} \to \mathbf{Cat}_\infty$, where $\mathbf{Cat}_\infty$ is the ∞-category of ∞-categories. This seems complicated, but luckily we don't need to go down this road.

It is important, though, to know that similarly to the classical categorical case (7.6), any map $\alpha : b_1 \to b_2$ in $\mathcal{B}$ induces an essentially unique functor $\alpha_! = \mathcal{C}_\alpha : \mathcal{C}_{b_1} \to \mathcal{C}_{b_2}$ defined by means of coCartesian lifts.

We're now almost ready to define a symmetric monoidal ∞-category, but first we need an elementary definition. Recall that $\mathcal{H}$ denotes the homotopy category of spaces.

Definition 8.7. [Lur09, 1.1.5.14] [Gro10, 1.35] A map $f : S \to T$ of simplicial sets is a *categorical equivalence* if the induced map $hf : hS \to hT$ is an equivalence of $\mathcal{H}$-enriched categories.

Equivalently, this means that $\mathfrak{C}(f) : \mathfrak{C}(S) \to \mathfrak{C}(T)$ is an equivalence of simplicial categories[11], i.e. the functor $\pi_0\mathfrak{C}(f) : \pi_0\mathfrak{C}(S) \to \pi_0\mathfrak{C}(T)$ is essentially surjective, and for all $x, y \in \mathfrak{C}(S)$ the map $\mathrm{Map}_{\mathfrak{C}(S)}(x, y) \to \mathrm{Map}_{\mathfrak{C}(T)}(\mathfrak{C}(f)(x), \mathfrak{C}(f)(y))$ is a weak equivalence of simplicial sets (i.e. it induces a weak homotopy equivalence of spaces after geometric realization). See [Lur09, Lemma 3.1.3.2] for conditions on f which are equivalent to being a categorical equivalence.

And finally, drawing inspiration from Corollary 7.11, we have:

Definition 8.8. A *symmetric monoidal ∞-category* is a coCartesian fibration $\mathcal{C} \to N(\Gamma^{\mathrm{op}})$ such that the functor

$$\mathcal{C}_{[n]} \xrightarrow{(\mathcal{C}_{\rho_1}, \ldots, \mathcal{C}_{\rho_n})} (\mathcal{C}_{[1]})^{\times n}$$

is a categorical equivalence for all $n \geq 0$.

[11] Also called a *Dwyer-Kan equivalence*.

Remark 8.9. There is likely an equivalent formulation of the sort: a symmetric monoidal ∞-category is an ∞-functor $N(\Gamma^{op}) \to \mathbf{Cat}_\infty$ satisfying that some Segal map is a categorical equivalence.[12]

We can now start doing higher algebra.

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[12]Following in the steps of Footnotes 6 and 10, we might say that ∞-special Γ-∞-categories model symmetric (weak) ∞-monoids in $\mathbf{Cat}_\infty$. Comparing to the case of E_∞-spaces and symmetric monoidal ∞-categories, one can be led to think that the former is weird, because we are considering 1-functors $\Gamma^{op} \to \mathbf{Top}$ and yet modeling ∞-monoids. The point is that homotopy coherent diagrams can be rectified, see [Lur09, 4.2.4.4], so 1-functors 1-functors $\Gamma^{op} \to \mathbf{Top}$ are equivalently ∞-functors $N(\Gamma^{op}) \to Sp$, where Sp is the ∞-category of spaces. This puts this example in a form more suitable for comparison with the other two cases.