Introduction to stable homotopy theory
Exercise sheet n° 3

1. i. Using the Puppe sequence, recover the long exact sequence for reduced cohomology groups with coefficients in an abelian group G associated to a cofiber sequence (in particular, to a Hurewicz cofibration).
   ii. Formulate the statement of exactness associated to the Puppe sequence for fiber sequences. Recover the long exact sequence for homotopy groups associated to a fiber sequence (in particular, to a Hurewicz fibration).
   iii. As for the reduced homology long exact sequence: you can also get it like this, but it's a bit more troublesome. There is a construction $SP : \mathrm{Top}_* \to \mathrm{Top}_*$ (the *infinite symmetric product*) that satisfies $\pi_*(SP(-)) \cong H_*$[1]; now, the SP functor takes cofiber sequences[2] to quasifibration sequences, which is something weaker than a fiber sequence but good enough to generate a long exact sequence in homotopy groups, so you're done. This is a lot of work, of course: this is the bulk of the Dold-Thom theorem. For more details, see [AGP02, Page 178].

2. Let X, Y be well-pointed spaces. Prove that the square

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & * \end{array}$$

   is homotopy cocartesian. Deduce that if X is p-connected and Y is q-connected, then the canonical map $X \vee Y \to X \times Y$ is $(p+q+1)$-connected.

3. i. Prove that a map $f : X \to Y$ is nullhomotopic if and only if it factors through $i : X \to CX$.
   ii. Let $X \xrightarrow{f} Y$ be a map. Prove that there is a bijection between the dotted maps making the following diagram commute

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & Cf \\ & & & \underset{g}{\searrow} & \vdots \\ & & & & Z \end{array}$$

   and nullhomotopies of gf. In other words, $i : Y \to Cf$ is the universal map from Y such that if is nullhomotopic[3]; the comparison map $Cf \to Z$ depends on a choice of nullhomotopy for gf.

4. Let X be a pointed CW-complex which is connected, non-contractible and acyclic. Acyclic means that $\widetilde{H}_*(X) = 0$. Such spaces exist [Hat02, 2.38]. Prove that ΣX is contractible. Deduce that $X \to * \to *$ is a cofiber sequence.

5. Prove that if $f : X \to Y$ is a weak equivalence then Ωf is a weak equivalence, and Σf is also a weak equivalence provided the spaces are well-pointed.

6. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a cofiber sequence.

---

[1] Think about this result: it tells you that you can subordinate the study of homology to that of homotopy! No need to use chain complexes at all. Indeed, that is the point of view of the algebraic topology book [AGP02], which focuses on homotopy theory from the get-go, and uses Dold-Thom to *define* homology.

[2] Where the left map is a cofibration, to be precise, which is always true up to equivalence, as we have already mentioned.

[3] Compare with the universal property of the strict cofiber $X \to Y \to Y/f(X)$.

i. Prove that if f is n-connected and X is m-connected, then there is a truncated long exact sequence of homotopy groups

$$\pi_{n+m}(X) \to \pi_{n+m}(Y) \to \pi_{n+m}(Z) \to \pi_{n+m-1}(X) \to \cdots .$$

ii. Prove that if f is n-connected and X is connected, then Z is n-connected.

iii. (*) Suppose X is simply connected. Prove that if Z is n-connected then f is n-connected. Note that Exercise 4 gives a counterexample in the absence of the simply-connectedness hypothesis.

7. Prove that if the square of pointed spaces

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is homotopy cartesian, then the induced maps of homotopy fibers are weak homotopy equivalences. Conversely, given a homotopy-commutative square as above, prove that if the induced map on homotopy fibers is a weak equivalence, then the square is homotopy cartesian.

8. i. Let X be a space and $U, V \subseteq X$ be open subsets such that $X = U \cup V$. Prove that the pushout square

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

is homotopy cocartesian. One can prove that every homotopy cocartesian square is equivalent to one of this form.

ii. (*) Let

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow[h]{} & Z \end{array}$$

be a homotopy pushout of well-pointed spaces. Construct the following *Mayer–Vietoris* long sequence in which every two consecutive maps is a cofiber sequence:

$$X \vee Y \xrightarrow{(h,k)} Z \xrightarrow{\delta} \Sigma W \xrightarrow{\Sigma f - \Sigma g} \Sigma X \vee \Sigma Y \xrightarrow{\Sigma h + \Sigma k} \Sigma Z \longrightarrow \cdots$$

Deduce a long exact sequence in reduced cohomology groups associated to this homotopy pushout[4]; in particular, deduce the Mayer–Vietoris long exact sequence in reduced cohomology groups.

iii. Prove that $\widetilde{H}^n(-;G)$ takes homotopy pushouts of well-pointed spaces into weak pullbacks of abelian groups. A *weak pullback* is like a pullback, only that the arrow in the universal property need not be unique.

References

[AGP02] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto. *Algebraic topology from a homotopical viewpoint*. Universitext. Springer-Verlag, New York, 2002. Translated from the Spanish by Stephen Bruce Sontz.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[4] This generalizes to a Bousfield–Kan spectral sequence computing the homology of a homotopy colimit.