Introduction to stable homotopy theory
Exercise sheet n° 4

0. Give an example proving that excision doesn't work for relative homotopy. That is: find an example of a CW-complex X decomposed as a union of two subcomplexes $X = A \cup B$ such that $\pi_n(X, B)$ is not isomorphic to $\pi_n(A, A \cap B)$.

1. Let $f : X \to Y$ be a morphism in CW_*. Prove that if $f_* : \widetilde{H}_*(X;\mathbb{Z}) \to \widetilde{H}_*(Y;\mathbb{Z})$ is an isomorphism, then $f_* : h_*(X) \to h_*(Y)$ is an isomorphism for any reduced homology theory h_*.

2. Let h_* be a reduced homology theory on CW_*.
   i. Prove that $h_*(*) = 0$.
   ii. Prove that for any subcomplex $A \subseteq X$ there is an associated long exact sequence.
   iii. Prove that there is a Mayer–Vietoris long exact sequence associated to a decomposition into subcomplexes $X = A \cup B$. (Hint: use the double mapping cylinder to obtain a decomposition of a homotopy equivalent space into homotopy equivalent open subsets.)
   iv. (*) Let $X_0 \to X_1 \to X_2 \to \dots$ be a sequence of sub-CW-complex inclusions. Prove that the natural map

   $$\mathrm{colim}_i h_n(X_i) \to h_n(\mathrm{colim}_i X_i)$$

   is an isomorphism. The analogous statement for cohomology is false in general: limits of abelian groups behave less well than colimits. While the functor $\mathrm{colim} : \mathrm{Fun}(\mathbb{N}, \mathrm{Ab}) \to \mathrm{Ab}$ is exact[1], the functor $\lim : \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Ab})$ is only left exact, and its right derived functor is called $\lim^1$. The *Milnor* $\lim^1$ *sequence* is a natural short exact sequence

   $$0 \longrightarrow \lim{}^1 h^{q-1}(X_i) \longrightarrow h^q(X) \longrightarrow \lim h^q(X_i) \longrightarrow 0.$$

   See [Hat02, 3F.8] or [Sel97, 13.1.3].
   v. Let h'_* be another reduced homology theory and let $T : h_* \Rightarrow h'_*$ be a morphism of homology theories. That means that it is a natural transformation that commutes with suspension. Suppose that $T(S^0) : h_*(S^0) \to h'_*(S^0)$ is an isomorphism. Deduce that T is a natural isomorphism. The analogous statement for cohomology is also true and can be proven similarly.

3. Prove that if E is an Ω-spectrum, then $E^n(-) := [-, E_n]$ defines a cohomology theory. Here we use the convention that $E_{-n} = \Omega^n E_0$ for $n > 0$.

4. Using the Brown representability theorem for functors $h^n : \mathrm{Ho}(CW_*^{\geq 1})^{\mathrm{op}} \to \mathrm{Ab}$, prove that if h^* is a homology theory, then there exists an Ω-spectrum E such that $h^* \cong E^*$ as cohomology theories $\mathrm{Ho}(CW_*)^{\mathrm{op}} \to \mathrm{GrAb}_{\mathbb{Z}}$ (note that the connectedness hypothesis is gone).[2]

5. Define the category CW^2 of *CW-pairs* to have as objects pairs (X, A) where X is a CW-complex and A is a subcomplex. Morphisms are continuous maps $X \to Y$ such that $f(A) \subseteq B$. If $A = \emptyset$ we omit it from the notation. Define a *(generalized, unreduced) homology theory* to be a sequence of functors $H_n : CW^2 \to \mathrm{Ab}$, $n \in \mathbb{Z}$, together with natural transformations $H_n(X, A) \to H_{n-1}(A)$, satisfying the following axioms:

[1] Because sequential colimits of abelian groups commute with finite limits.

[2] You may want to use the result that $F(X, Y)$ is homotopy equivalent to a CW-complex (and not merely weakly homotopy equivalent) if X, Y are CW-complexes and X is finite [Mil59]; this applies in particular to loop spaces.

- Homotopy: if $f, g : (X, A) \to (Y, B)$ are homotopic (via a homotopy of pairs $(X \times I, A \times I) \to (Y, B)$), then $H_n(f) = H_n(g)$.
- Exactness: any CW-pair (X, A) yields a long exact sequence of abelian groups

$$\cdots \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

- Excision: If X is the union of subcomplexes A and B, then the inclusion $(A, A \cap B) \to (X, B)$ induces an isomorphism

$$H_*(A, A \cap B) \to H_*(X, B).$$

- Additivity: If $\{(X_i, A_i)\}_i$ are CW-pairs, the canonical map $\bigoplus_i H_*(X_i, A_i) \to H_*(\bigsqcup X_i, \bigsqcup A_i)$ is an isomorphism.[3]

i. For an unreduced homology theory H_*, prove:
   a) If (Y, X) is a CW-pair with inclusion $i : X \to Y$, then

$$H_*(Y, X) \xrightarrow{\cong} H_*(Ci, *) \cong H_*(Y/X, *).$$

   b) $H_*(X) \cong H_*(X, *) \oplus H_*(*)$ naturally in X.
   c) Prove that H_* determines a reduced homology theory on CW_* by $\widetilde{H}_*(X) = H_*(X, *)$.

ii. Prove that a reduced homology theory $\widetilde{H}_*$ on CW_* determines an unreduced homology theory H_* on CW^2 by $H_*(X) = \widetilde{H}_*(X_+)$, and $H_*(X, A) = \widetilde{H}_*(X/A)$ for $A \neq \emptyset$.

iii. Conclude that the categories of reduced and unreduced homology theories are equivalent.

References

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Mil59] John Milnor. On spaces having the homotopy type of a CW-complex. *Trans. Amer. Math. Soc.*, 90:272–280, 1959.

[Sel97] Paul Selick. *Introduction to homotopy theory*, volume 9 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1997.

[3] We can make a similar definition in the category of all pairs of spaces. We'd then add a weak equivalence axiom, saying that weak equivalences of pairs get mapped to isomorphisms, in the exactness axiom we'd take a cofiber sequence, and in the excision axiom we'd decompose X as the union of the interiors of two subspaces; additivity is analogous.