Introduction to stable homotopy theory
Exercise sheet n° 6

1. Let $\mathcal{C}$ be a cocomplete category with a set of maps $\mathcal{J}$.
   i. Prove that the coproduct of cell complexes is a cell complex.
   ii. Let $X \to Y$ and $Y \to Z$ be relative cell complexes. Prove that the composition is a relative cell complex.
   iii. Let $X_0 \to X_1 \to \cdots$ be a sequential diagram in $\mathcal{C}$ where all the arrows are relative cell complexes. Prove that the natural map $X_0 \to \mathrm{colim}_n X_n$ is a relative cell complex. In particular, if $X_0 = *$ or X_0 is a cell complex, then $\mathrm{colim}_n X_n$ is a cell complex.
   iv. Prove that a pushout of a relative cell complex along any map is a relative cell complex.

2. Let $\pi_*^s := \pi_*^s(S^0)$ be the $\mathbb{N}$-graded abelian group of stable homotopy groups of spheres.
   i. Prove that the twist $S^1 \wedge S^1 \to S^1 \wedge S^1$ has degree -1. Deduce that the twist $S^n \wedge S^m \to S^m \wedge S^n$ has degree $(-1)^{nm}$.
   ii. Let $\alpha \in \pi_n^s, \beta \in \pi_m^s$. Choose representatives $[f : S^{n+k} \to S^k] \in \pi_n^s$ and $[g : S^{m+k} \to S^k] \in \pi_m^s$. Prove that the operation $* : \pi_n^s \times \pi_m^s \to \pi_{n+m}^s$ given by
   $$\alpha * \beta = [f \circ \Sigma^n g : S^{m+n+k} \to S^k] \in \pi_{n+m}^s$$
   is well-defined and associative.
   iii. Prove that if k is even, then
   $$[f \circ \Sigma^n g : S^{m+n+k} \to S^k] = [f \wedge g : S^{n+m+2k} \to S^{2k}]$$
   as elements of π_{n+m}^s. What happens if k is odd?
   iv. Prove that $*$ is graded-commutative. That is,
   $$[f] * [g] = (-1)^{nm}[g] * [f]$$
   for $[f] \in \pi_n^s, [g] \in \pi_m^s$.
   v. Finally, prove that $*$ is distributive with respect to $+$, and so π_*^s is a graded-commutative graded ring. As a commentary: Each element of π_n^s, $n \geq 1$ is nilpotent, i.e. some power of it is zero.[1] This is a theorem of Nishida (1973), see [Rav92].

3. i. (*) Prove that if A is a cell spectrum and $f : B \to B'$ is a weak equivalence of Ω-spectra, then $f_* : [A, B] \to [A, B']$ is a bijection. Dually, prove that if $f : A \to A'$ is a weak equivalence of cell spectra and B is an Ω-spectrum, then $f^* : [A', B] \to [A, B]$ is a bijection.[2]
   ii. Prove that $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y) \cong [QX, RY]$.
   iii. If X is a cell spectrum, prove that $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y) \cong [X, RY]$.
   iv. If Y is an Ω-spectrum, prove that $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y) \cong [QX, Y]$.
   v. If X is a cell spectrum and Y is an Ω-spectrum, prove that $\mathrm{Hom}_{\mathrm{Ho(Sp)}}(X, Y) \cong [X, Y]$.

4. Let $h\mathrm{Sp}$ be the category whose objects are the cell Ω-spectra and whose maps $X \to Y$ are given by $[X, Y]$. Prove that $h\mathrm{Sp}$ and $\mathrm{Ho(Sp)}$ are equivalent but not isomorphic. This $h\mathrm{Sp}$ together with $\delta : \mathrm{Sp} \to \mathrm{Ho(Sp)}$, $X \mapsto QRX$, $\delta(f) = [QRf]$ satisfies the weaker universal property of localization explained in Side remark 4.2 in the course.

References

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[MP12] J. P. May and K. Ponto. *More concise algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories.

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

[1] So good luck trying to describe π_*^s in terms of generators and relations!

[2] See [Hov99, 1.2.5.(iv)] or [MP12, 14.3.14, 14.4.8] for abstract proofs.