Introduction to stable homotopy theory
Exercise sheet n° 7

1. (Splitting lemma) Let $f : X \to Y$ be a map of spectra. Suppose there exists a map $s : Y \to X$ such that $f \circ s : Y \to Y$ is a weak equivalence. Let $i : F \to X$ be the homotopy fiber of f. Prove that the map $(i, s) : F \vee Y \to X$ is a weak equivalence.

2. This exercise proves that levelwise Hurewicz cofibrations satisfy some properties which are also satisfied by the cofibrations (retracts of relative cell spectra).
   i. Let $f : X \to Y$ be a map of spectra which is a levelwise Hurewicz cofibration. Observe that the pushout of f along another map is also a levelwise Hurewicz cofibration, and prove that if moreover f is a weak equivalence, then its pushout is so too.
   ii. (Gluing lemma) Prove that given a commutative diagram of spectra as below, where the arrows $X \to Y$ and $X' \to Y'$ are levelwise Hurewicz cofibrations and the vertical maps are weak equivalences
   $$\begin{array}{ccccc} Z & \longleftarrow & X & \rightarrowtail & Y \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ Z' & \longleftarrow & X' & \rightarrowtail & Y' \end{array}$$
   the induced map of pushouts is a weak equivalence.
   iii. If $X^0 \to X^1 \to X^2 \to \cdots$ is a sequence of levelwise Hurewicz cofibrations, prove that $\pi_n(\mathrm{colim}_i X^i) \cong \mathrm{colim}_i \pi_n(X^i)$. If moreover the maps in the sequence are all weak equivalences, then prove that the canonical map $X^0 \to \mathrm{colim}_i X^i$ is a weak equivalence.
   iv. Use the above to prove the following: if $A \in \mathrm{Top}_*$ is a pointed cell complex, then the functor $- \wedge A : \mathrm{Sp} \to \mathrm{Sp}$ preserves weak equivalences.