ERRATA TO "THE COTANGENT COMPLEX AND THOM SPECTRA"

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We correct the definition of the indecomposables functor and the proof of Lemma 3.30 in [RS20]. The statements of Lemma 3.30, Remark 3.31, Theorem 3.32, and Corollary 3.34 hold with the corrected definition. We also give a shorter proof of Theorem 3.32 and correct a formula in Remark 3.33. These corrections concern Section 3.3 only. The numbering of statements below refers to [RS20].

We thank Ben Steffan for pointing out the mistakes in Lemma 3.30.

0.1. **The main problem and how to fix it.** Fix an E_∞-ring spectrum B. We use the notations

$$I_0 : \mathrm{CAlg}_{B//B} \xrightarrow{\simeq} \mathrm{NUCA}_B, \qquad U : \mathrm{NUCA}_B \to \mathrm{Mod}_B, \qquad I \simeq U \circ I_0$$

from Remark 3.6, and write $F : \mathrm{Mod}_B \to \mathrm{NUCA}_B$ for the left adjoint of U.

Definition 3.29 incorrectly identifies derived indecomposables with the cofiber of binary multiplication in Mod_B. Denote that construction by

$$Q_{\mathrm{bin}}(N) = \mathrm{cofib}_{\mathrm{Mod}_B}(UN \wedge_B UN \xrightarrow{\mu} UN).$$

For the trivial nonunital algebra $ZM = I_0(B \oplus M)$, multiplication is nullhomotopic, so

$$Q_{\mathrm{bin}}(ZM) \simeq M \oplus \Sigma(M \wedge_B M),$$

not M in general, as claimed in the proof of Lemma 3.30(1). Moreover, an equivalence $QZM \simeq M$, even if it existed, would not by itself establish the terminal-object property in $(Q \downarrow M)$ asserted there.

The problem affects the definition, rather than just this computation. For example, let $B = H\mathbb{Q}$ and $N = F(B)$, so that N is the nonunital polynomial algebra on one degree-zero generator and $UN \simeq H(x\mathbb{Q}[x])$. Then

$$\pi_1 Q_{\mathrm{bin}}(N) \cong \ker(x\mathbb{Q}[x] \otimes_{\mathbb{Q}} x\mathbb{Q}[x] \xrightarrow{\mu} x\mathbb{Q}[x]) \neq 0,$$

since the kernel contains $x \otimes x^2 - x^2 \otimes x$. Thus $Q_{\mathrm{bin}}F(B) \not\simeq B$, contrary to Lemma 3.30(2). The cofiber construction must be replaced by derived indecomposables, which we characterize by their adjunction.

Corrected Definition 3.29. Let

$$Z : \mathrm{Mod}_B \to \mathrm{NUCA}_B, \qquad M \mapsto I_0(B \oplus M),$$

where $B \oplus M$ is the canonical trivial square-zero extension, regarded as an object of $\mathrm{CAlg}_{B//B}$; see [Lur17, Rmks. 7.3.4.15, 7.3.4.16]. The *indecomposables functor* $Q : \mathrm{NUCA}_B \to \mathrm{Mod}_B$ is the left adjoint of Z, whose existence is proved below. Equivalently, it is characterized by natural equivalences

$$\mathrm{Map}_{\mathrm{Mod}_B}(QN, M) \simeq \mathrm{Map}_{\mathrm{NUCA}_B}(N, ZM).$$

Lemma 3.30. *(1) The functor Z admits a left adjoint Q. Moreover, $UZ \simeq \mathrm{id}_{\mathrm{Mod}_B}$, and the multiplication of ZM is nullhomotopic for every $M \in \mathrm{Mod}_B$.*

(2) $Q \circ F \simeq \mathrm{id}_{\mathrm{Mod}_B}$.

(3) Up to equivalence, Z is the unique functor $Z' : \mathrm{Mod}_B \to \mathrm{NUCA}_B$ with $UZ' \simeq \mathrm{id}_{\mathrm{Mod}_B}$.

Proof. (1) The identity $UZ \simeq \mathrm{id}_{\mathrm{Mod}_B}$ and the assertion about multiplication follow from the construction of the trivial square-zero extension. The ∞-categories Mod_B and NUCA_B are presentable, and U reflects small limits and sifted colimits; see [Lur17, Cor. 3.2.2.4, 3.2.3.5, Prop. 3.2.3.1], applied to nonunital commutative algebras. Since $UZ \simeq \mathrm{id}_{\mathrm{Mod}_B}$, applying U to the canonical comparison maps shows that Z preserves small limits and filtered colimits. In particular, Z is accessible. The adjoint functor theorem [Lur09, Cor. 5.5.2.9(2)] therefore supplies a left adjoint Q.

(2) The composite QF is left adjoint to $UZ \simeq \mathrm{id}_{\mathrm{Mod}_B}$, and is consequently equivalent to the identity.

(3) Suppose that $UZ' \simeq \mathrm{id}_{\mathrm{Mod}_B}$. Since U reflects limits, Z' preserves finite limits. For a finite-limit-preserving functor, the derivative is obtained by applying the functor to spectrum objects; see Section 2(4). Hence

$$\partial U \circ \partial Z' \simeq \mathrm{id}_{\mathrm{Sp}(\mathrm{Mod}_B)} \simeq \partial U \circ \partial Z.$$

By Remark 3.6 and Theorem 3.7,

$$\partial U : \mathrm{Sp}(\mathrm{NUCA}_B) \to \mathrm{Sp}(\mathrm{Mod}_B)$$

is an equivalence: $I \simeq U \circ I_0$, while both ∂I and ∂I_0 are equivalences. It follows that $\partial Z' \simeq \partial Z$. Evaluation at level zero commutes with applying a finite-limit-preserving functor to spectrum objects, so

$$\begin{aligned} Z' \circ \Omega^\infty_{\mathrm{Mod}_B} &\simeq \Omega^\infty_{\mathrm{NUCA}_B} \circ \partial Z' \\ &\simeq \Omega^\infty_{\mathrm{NUCA}_B} \circ \partial Z \simeq Z \circ \Omega^\infty_{\mathrm{Mod}_B}. \end{aligned}$$

Since Mod_B is stable, $\Omega^\infty_{\mathrm{Mod}_B}$ is an equivalence, and therefore $Z' \simeq Z$. □

0.2. **An alternative proof of Thm. 3.32.** Theorem 3.32 holds for the corrected Q. Its proof can be replaced by the arguably simpler argument we present below.

Remark. Remark 3.31 remains valid; its final identification now follows directly from the definition $Z(M) = I_0(B \oplus M)$, without appealing to Lemma 3.30.

For the following proof, let $K_B : \mathrm{Mod}_B \to \mathrm{CAlg}_{B//B}$ be $(B \vee -) \circ Z$, so the functor K from Remark 3.31 is equivalent to K_B followed by restriction $\mathrm{CAlg}_{B//B} \to \mathrm{CAlg}_{A//B}$.

Theorem 3.32. *Let $B \in \mathrm{CAlg}$. The following diagram commutes up to natural equivalence:*

$$\begin{array}{ccc} \mathrm{CAlg}_{B//B} & \xrightarrow{\Sigma^\infty} & \mathrm{Sp}(\mathrm{CAlg}_{B//B}) \\ \downarrow{\scriptstyle I_0} & & \downarrow{\scriptstyle \Omega^\infty_{\mathrm{Mod}_B} \circ \partial I} \\ \mathrm{NUCA}_B & \xrightarrow[Q]{} & \mathrm{Mod}_B. \end{array}$$

Proof. Put $\mathcal{C} = \mathrm{CAlg}_{B//B}$ and let

$$E = \Omega^\infty_{\mathrm{Mod}_B} \circ \partial I : \mathrm{Sp}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Mod}_B$$

be the equivalence of Theorem 3.7. The construction of the trivial square-zero extension gives

$$K_B \simeq \Omega^\infty_{\mathcal{C}} \circ E^{-1};$$

see [Lur17, Rmks. 7.3.4.15, 7.3.4.16]. Consequently, $E \circ \Sigma^\infty_{\mathcal{C}}$ is left adjoint to K_B. On the other hand, $Q \circ I_0$ is left adjoint to $I_0^{-1} \circ Z \simeq K_B$. Uniqueness of left adjoints gives

$$Q \circ I_0 \simeq E \circ \Sigma^\infty_{\mathcal{C}} = \Omega^\infty_{\mathrm{Mod}_B} \circ \partial I \circ \Sigma^\infty_{\mathcal{C}},$$

as required. □

Remark. In Remark 3.33, the formula expressing $P_1(U \circ F)$ in terms of the derivative should read

$$P_1(U \circ F) \simeq \Omega^\infty_{\mathrm{Mod}_B} \circ \partial(U \circ F) \circ \Sigma^\infty_{\mathrm{Mod}_B},$$

with Σ^∞ and Ω^∞ in this order. The equivalence $P_1(U \circ F) \simeq \mathrm{id}_{\mathrm{Mod}_B}$ and its interpretation in that remark are unchanged.

0.3. **The functor Z is not fully faithful.** In the previous, wrong proof of Lemma 3.30(1), we used that $QZM \simeq M$. This is not correct: the counit of the $Q \dashv Z$ adjunction need not be an equivalence, i.e. Z is not fully faithful.

For an explicit example, take $B = M = H\mathbb{Q}$, and let $\alpha : FB \to FB$ send the generator t to the square x^2 of the generator of the target. There is an equivalence

$$\mathrm{cofib}_{\mathrm{NUCA}_B}(\alpha) \simeq ZB.$$

Indeed, adjoining units identifies this cofiber with the pushout of augmented commutative B-algebras

$$B[x] \wedge_{B[t]} B \simeq H(\mathbb{Q}[x]/(x^2)) \simeq B \oplus B,$$

where the structure maps send t to x^2 and 0, respectively, and the last term is the trivial square-zero extension. Here $B[x]$ and $B[t]$ are free commutative B-algebras on degree-zero generators. The pushout has no higher homotopy groups because $\mathbb{Q}[x]$ is free as a $\mathbb{Q}[t]$-module, with basis $\{1, x\}$, under $t \mapsto x^2$.

Since Q preserves colimits and $QFB \simeq B$, applying Q gives

$$QZB \simeq \mathrm{cofib}_{\mathrm{Mod}_B}(B \xrightarrow{Q\alpha} B).$$

Under these identifications, $Q\alpha$ is zero: its adjoint $FB \to ZB$ sends the generator to a square, which vanishes in ZB. Consequently,

$$QZB \simeq B \oplus \Sigma B.$$

The counit is the projection $B \oplus \Sigma B \to B$, so it is not an equivalence. In particular, Z is not fully faithful.

0.4. **Comments on the literature.**

0.4.1. *Basterra's model categorical approach.* The adjunction $Q \dashv Z$ is an ∞-categorical analogue of the derived adjunction constructed immediately after [Bas99, Prop. 3.1]. Write $Q_{\mathrm{pt}} \dashv Z_{\mathrm{pt}}$ for Basterra's underived adjunction. Since Q_{pt} is defined by an ordinary pushout of modules, its counit $Q_{\mathrm{pt}}Z_{\mathrm{pt}}M \to M$ is an isomorphism. This does not imply the counit of the derived adjunction on homotopy categories is an isomorphism, and the same example as above works; computing $(\mathbb{L}Q_{\mathrm{pt}})(\mathbb{R}Z_{\mathrm{pt}}M)$ requires applying Q_{pt} to a cofibrant replacement of $Z_{\mathrm{pt}}M$ in the category of nucas.

0.4.2. *Operadically.* The adjunction $Q \dashv Z$ is a special case of the construction described in [BM25, Section 4.1]. Let $\mathcal{O}$ be the nonunital commutative operad in Mod_B, and let $\mathbf{1}$ be the operad with $\mathbf{1}(1) = B$ and $\mathbf{1}(n) = 0$ for $n \neq 1$. The canonical augmentation $\varepsilon : \mathcal{O} \to \mathbf{1}$ induces a free–forgetful adjunction

$$\varepsilon_! : \mathrm{NUCA}_B \rightleftarrows \mathrm{Mod}_B : \varepsilon^*.$$

Restriction along ε equips a module with trivial operations of arity at least two, so $\varepsilon^* \simeq Z$. Consequently, $\varepsilon_! \simeq Q$ by uniqueness of left adjoints. Thus our functors Q and Z are their cotangent fibre functor cot and square-zero functor sqz, respectively.

Their description of the resulting comonad on page 41 gives

$$QZM \simeq \bigoplus_{n \geq 1} (\mathrm{Bar}(\mathcal{O})(n) \wedge_B M^{\wedge_B n})_{h\Sigma_n},$$

where $\mathrm{Bar}(\mathcal{O})$ is the operadic bar cooperad and $(-)_{h\Sigma_n}$ denotes homotopy orbits. The counit is the projection onto the arity-one summand M. The higher-arity summands need not vanish: for instance, when $B = M = H\mathbb{Q}$, the arity-two summand is $\Sigma H\mathbb{Q}$. This also explains why Z is not fully faithful.

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