On higher and iterated topological Hochschild homology

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December 8, 2017

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Brave new algebra: replace $\mathbb{Z}$ with the sphere spectrum $\mathbb{S}$, and $HH^{\mathbb{Z}}(R)$ by $THH^{\mathbb{S}}(R) = THH(R)$. Get a topological trace map

$$\begin{array}{ccc} K(R) & \xrightarrow{\quad tr \quad} & HH^{\mathbb{Z}}(R) \\ & {\scriptstyle tr}\searrow \quad \nearrow & \\ & THH(R) & \end{array}$$

which exists for any ring spectrum R.

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$$[n] \mapsto R^{\wedge(n+1)},$$

$$d_i(a_0 \wedge \cdots \wedge a_n) = a_0 \wedge \cdots \wedge a_i a_{i+1} \wedge \cdots \wedge a_n \quad i \neq n,$$

$$d_n(a_0 \wedge \cdots \wedge a_n) = a_n a_0 \wedge a_1 \wedge \cdots \wedge a_{n-1}.$$

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This gives a simplicial spectrum $B^{\mathrm{cy}}_\bullet(R)$ whose geometric realization is a spectrum $THH(R)$.

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There is also "higher THH". Generalizes Pirashvili's *higher order Hochschild homology* and is related to topological André-Quillen homology.

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for a discrete ring A, where $R[-] = R \wedge_{\mathbb{S}} \Sigma^{\infty}_{+}(-)$, together with their graded multiplications.

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In **Part 2**: complete identification of $THH(KU)$, $T^n \otimes KU$ and $S^n \otimes KU$ as commutative KU-algebras.Two descriptions: one as $KU[G]$ where G is some product of Eilenberg-Mac Lane spaces, and one as a free commutative KU-algebra on a rational KU-module.

Introduction

Graded multiplications on iterated bar constructions

Higher and iterated THH of KU

Questions raised

Instances of the *bar construction* BA:

- classifying space of a topological monoid A,
- $HH^k(A, k)$ of an augmented k-algebra A,
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Goal: describe a framework which unifies these constructions. Find conditions on A such that $\{B^n A\}_{n \geq 0}$ gets a graded multiplication, and identify it.

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Want: symmetric monoidal geometric realization $|-| : s\mathcal{V} \to \mathcal{V}$, to have an induced $B_{\mathcal{V}} = |B_\bullet| : \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \to \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}}$.

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Theorem (S.)

Let $F : s\mathbf{Set} \to \mathcal{V}$ be a symmetric monoidal functor which is a left adjoint. Let $\Delta^\bullet$ be the canonical cosimplicial simplicial set. Then

$$|-|_{\mathcal{V}} := - \otimes_{\Delta} F\Delta^\bullet : s\mathcal{V} \to \mathcal{V}$$

is symmetric monoidal.

Theorem (S.)

Let $s\mathbf{Set} \xrightarrow{F} \mathcal{V} \xrightarrow{G} \mathcal{W}$ *be symmetric monoidal functors which are left adjoints. Then*

$$|-|_{\mathcal{V}} = - \otimes_{\Delta} F\Delta^{\bullet} : s\mathcal{V} \to \mathcal{V}$$

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are symmetric monoidal, and there is a monoidal isomorphism

$$s\mathcal{V} \underset{G|-|_{\mathcal{V}}}{\overset{|G-|_{\mathcal{W}}}{\rightrightarrows}} \mathcal{W}, \quad \Downarrow\cong$$

We set $B_{\mathcal{V}} = |B_\bullet|_{\mathcal{V}} : \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}} \to \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}}$ and similarly for $\mathcal{W}$.

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Corollary

Let $s\mathbf{Set} \xrightarrow{F} \mathcal{V} \xrightarrow{G} \mathcal{W}$ *be symmetric monoidal functors which are left adjoints. There is an isomorphism in* $\mathbf{CMon}(\mathcal{W})^{\mathrm{aug}}$

$$B_{\mathcal{W}} G(A) \cong G B_{\mathcal{V}}(A)$$

natural in $A \in \mathbf{CMon}(\mathcal{V})^{\mathrm{aug}}$.

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Now $B_{\mathcal{V}}$ is an endofunctor, so we can iterate it.

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$$B^*_{\mathcal{V}} : \mathbf{Ab}(\mathcal{V}) \to \mathbf{GrAb}(\mathcal{V}), \quad A \mapsto \{B^n A\}_{n \in \mathbb{N}}$$

and similarly for $B^*_{\mathcal{W}}$.

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$B^*_{\mathcal{V}}$ extends to $B^*_{\mathcal{V}} : \mathbf{Ring}(\mathcal{V}) \to \mathbf{GrRing}(\mathcal{V})$

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natural in $A \in \mathbf{Ring}(\mathcal{V})$.

Remark

If $s\mathbf{Set} \xrightarrow{F} \mathcal{V} \xrightarrow{G} \mathcal{W}$ are symmetric monoidal functors between symmetric monoidal categories, there is an induced sequence

$$s\mathbf{Set} \xrightarrow{F} \mathbf{CoComon}(\mathcal{V}) \xrightarrow{G} \mathbf{CoComon}(\mathcal{W})$$

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Example

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R: commutative ring spectrum. Let
$R[-] = R \wedge_{\mathbb{S}} \Sigma^{\infty}_{+}(-)$. $s\mathbf{Set} \xrightarrow{|-|} \mathbf{Top} \xrightarrow{R[-]} R\text{-}\mathbf{Mod}$ gives rise to the sequence

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$B^*_{\mathbf{Top}} : \mathbf{Ring}(\mathbf{Top}) \to \mathbf{GrRing}(\mathbf{Top})$ takes a discrete ring A to

$$\{K(A, n)\}_{n \geq 0}$$

with the graded multiplication constructed by Ravenel and Wilson (1980), representing the cup product in cohomology with A-coefficients.

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$B^*_{R\text{-}\mathbf{CoCoalg}} : \mathbf{Ring}(R\text{-}\mathbf{CoCoalg}) \to \mathbf{GrRing}(R\text{-}\mathbf{CoCoalg})$ takes T to the higher reduced THH

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$$THH^{R,[*]}(R[A],R) \cong R[K(A,*)]$$

in $\mathbf{GrRing}(R\text{-}\mathbf{CoCoalg})$.

Corollary

If A is a discrete ring and E is a commutative ring spectrum with a Künneth isomorphism, then

$$E_*(THH^{[*]}(\mathbb{S}[A], \mathbb{S})) \cong E_*(K(A, *))$$

in $\mathbf{GrRing}(\pi_*(E)\text{-}\mathbf{CoCoalg})$*, i.e. as* $\pi_*(E)$*-Hopf rings (coalgebraic rings).*

R: commutative ring spectrum. X: space. There exists a commutative ring spectrum $X \otimes R$ characterized by

$$\mathbb{S}\text{-}\mathbf{CAlg}(X \otimes R, A) \cong \mathbf{Top}(X, \mathbb{S}\text{-}\mathbf{CAlg}(R, A)).$$

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Theorem (McClure-Schwänzl-Vogt '97)

$THH(R) \cong S^1 \otimes R$ as commutative R-algebras.

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- R: Thom spectrum, X arbitrary. Schlichtkrull '11.

KU: complex topological K-theory commutative ring spectrum.

Goal: describe $X \otimes KU$ as a commutative KU-algebra, for any based space X. We describe $T^n \otimes KU$ and $S^n \otimes KU$.

Step 1: *THH*(*KU*). First description.

Theorem (Snaith '79)

There is a weak equivalence of commutative ring spectra

$$KU \simeq \mathbb{S}[\mathbb{C}P^\infty][x^{-1}]$$

for $x \in \pi_2\mathbb{S}[\mathbb{C}P^\infty]$.

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We first prove:

Theorem (Loday? (HH.) - S.)

Let R *be a commutative ring spectrum and* $x \in \pi_* R$. *There is a weak equivalence of commutative* $R[x^{-1}]$*-algebras*

$$THH(R[x^{-1}]) \simeq THH(R)[x^{-1}].$$

Proposition (Hesselholt-Madsen '97?)

If G is a topological abelian group, there is an isomorphism of commutative $\mathbb{S}[G]$-algebras

$$THH(\mathbb{S}[G]) \cong \mathbb{S}[G] \wedge \mathbb{S}[BG].$$

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Proven using the cyclic bar construction definition for THH and $B^{\mathrm{cy}} G \cong G \times BG$.

Corollary

There is an equivalence of commutative $\mathbb{S}[G][x^{-1}]$-algebras $THH(\mathbb{S}[G][x^{-1}]) \simeq \mathbb{S}[G][x^{-1}] \wedge \mathbb{S}[BG]$.

Theorem (S.)

There is a weak equivalence of commutative KU-algebras

$$THH(KU) \simeq KU \wedge \mathbb{S}[B\mathbb{C}P^{\infty}] = KU[K(\mathbb{Z}, 3)].$$

$THH(KU)$, second description:

$F : KU\text{-}\mathbf{Mod} \to KU\text{-}\mathbf{CAlg}$: free commutative algebra functor.

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Previously: $THH(L) \simeq L \vee \Sigma L_{\mathbb{Q}}$ additively (McClure-Staffeldt '93).

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Key lemma

$KU \wedge K(\mathbb{Z}, 3) \simeq \Sigma KU_{\mathbb{Q}}$.

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Key lemma

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Proof ingredients: Ravenel-Wilson's computation of the $K(1)$-homology of E-M spaces + $K(\mathbb{Z}, 3)_{\mathbb{Q}} \simeq S^3_{\mathbb{Q}}$ + Bott periodicity.

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Additionally: $T^n \otimes KU \simeq F\left(\bigvee_{i=1}^{n} (S^i)^{\vee \binom{n}{i}} \wedge KU_{\mathbb{Q}}\right)$.

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More generally, $\Sigma Y \otimes KU \simeq F(\Sigma Y \wedge KU_{\mathbb{Q}})$ for Y based CW-complex.

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We can deduce:

Theorem (S.)

$TAQ(KU) \simeq KU_{\mathbb{Q}}$ as KU-modules.

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- $X_0 \otimes KU \simeq Y_0 \otimes KU$.

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R: commutative ring spectrum. X, Y: spaces.

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When does $\Sigma X \simeq \Sigma Y$ imply $X \otimes R \simeq Y \otimes R$?

Examples

- Valid for all $\Sigma X \simeq \Sigma Y$ when $R =$ Thom spectrum (Schlichtkrull '11).

Let $X_0 = T^n$, $Y_0 = \bigvee_{i=1}^{n} (S^i)^{\vee \binom{n}{i}}$. Then $\Sigma X_0 \simeq \Sigma Y_0$, and:

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Dundas-Tenti '16: example of R with $X_0 \otimes R \not\simeq Y_0 \otimes R$.

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Goal: Understand why does KU behave like a Thom spectrum with respect to $X \otimes -$ ($X = T^n, S^n$). Investigate whether this formula gives the right result for $X \otimes KU$ for other X.

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HKR for non-connective algebras?

Thank you for your attention.